Constructions of Non Commutative Instantons on T^4 and K_3

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Abstract

We generalize the spectral-curve construction of moduli spaces of instantons on $\mathbf{T}^4$ and K_3 to noncommutative geometry. We argue that the spectral-curves should be constructed inside a twisted $\mathbf{T}^4$ or K_3 that is an elliptic fibration without a section. We demonstrate this explicitly for T^4 and to first order in the noncommutativity, for K_3. Physically, moduli spaces of noncommutative instantons appear as moduli spaces of theories with $\mathcal{N} = 4$ supersymmetry in 2+1D. The spectral curves are related to Seiberg-Witten curves of theories with $\mathcal{N} = 2$ in 3+1D. In particular, we argue that the moduli space of instantons of $U(q)$ Yang-Mills theories on a noncommutative K_3 is equivalent to the Coulomb branch of certain 2+1D theories with $\mathcal{N} = 4$ supersymmetry. The theories are obtained by compactifying the heterotic little-string theory on T^3 with global twists. This extends a previous result for noncommutative instantons on $\mathbf{T}^4$. We also briefly discuss the instanton equation on generic curved spaces.

Contents

1   Introduction

Since a realization of noncommutative gauge theories from string theory has been established in [1, 2], there has been a revived interest in the field. In [3], noncommutative gauge theories were obtained by studying D-branes with a large and constant NSNS B-field (in string units) and in [4, 5, 3], the moduli space of instantons on a flat noncommutative $\mathbf{R}^4$ was studied and was shown to be a smooth space. In later developments, the perturbative structure of various noncommutative field theories was studied and was shown to have a rich and surprising IR behavior [6, 7, 8] and the twistor construction of instanton moduli spaces has been extended to various noncommutative spaces [9].

In this work we will study the analytic structure of the moduli spaces of instantons on elliptically fibered noncommutative $\mathbf{T}^4$ and K_3. We will generalize the "spectral-curve" construction for the commutative case [10, 11] to the noncommutative geometries. We will see that, as far as the holomorphic structure of the moduli space of instantons goes, the essence of the noncommutativity can be captured by constructing the spectral curves in a modified elliptic fibration that does not have a section.

Aside from the theoretical interest, the moduli spaces of instantons on noncommutative spaces have important applications for M(atrix)-theory [12]. The M(atrix)-models for the 5+1D $(2,0)$ theories are given by quantum mechanics on moduli spaces of instantons on $\mathbf{R}^4$ [13, 14] and the M(atrix)-models of 3+1D $\mathcal{N} = 4$ SYM theories are derived from moduli

spaces of instantons on $\mathbf{T}^4$ [15]. These moduli spaces are singular but a regularization of the moduli space of instantons was suggested in [16] and was later shown to be equivalent to the moduli spaces of instantons on a noncommutative $\mathbf{R}^4$ [4, 5]. The special limit of $\mathcal{N} = 4$ SYM theories requires only the holomorphic structure of the moduli space of instantons [15], and hence our results for the moduli spaces of noncommutative instantons are directly applicable to the study of the M(atrix)-models of gauge theories.

Another application of moduli spaces of noncommutative instantons is to the construction of 2+1D $\mathcal{N} = 4$ moduli spaces of various compactified field-theories. Dualities in string theory and M-theory allow one to solve the quantum-corrected moduli space of many 2+1D gauge theories with 8 supersymmetries ($\mathcal{N} = 4$ in 2+1D) [17, 18]. The argument is usually an adaptation of the reasoning of [19] as follows. The gauge theory appears in a weak coupling limit of a certain string/M-theory background. "Weak coupling" means that the Planck scale M_p is infinite, in order to decouple gravity. Duality can relate this background to another one that is strongly-coupled. If we can use supersymmetry to argue that the coupling-constant of the strongly-coupled system has a factor λ that decouples from the moduli space in question, then we can solve the problem for $\lambda = 0$. At this value, the system is certainly not described by a gauge-theory any more. Nevertheless, the vacuum structure, namely the moduli-space, is still the same. The supersymmetry argument that one uses is the decoupling of hyper-multiplets from vector-multiplets. If λ is part of a hypermultiplet and we are interested in the vector-multiplet moduli space, we can safely argue that we can set λ to any value we need.

The Coulomb branch moduli space of a large class of 2+1D gauge-theories was found in [17]. They studied quiver gauge-theories [20], that is, theories with gauge groups of the form $\prod_{i=1}^{r+1} SU(a_i N)$ (where a_i are positive integers) and bi-fundamental hypermultiplets in representations $(\overline{\mathbf{a_i N}}, \mathbf{a_j N})$ such that the content and the a_i's are derived from an extended Dynkin diagram of ADE type [20]. They argued that the moduli space is equivalent to the moduli space of instantons on $\mathbf{R}^4$ with the gauge group corresponding to the particular Dynkin diagram and N is the instanton number. Other 2+1D theories were studied in [21].

A larger class of 2+1D questions about $\mathcal{N} = 4$ moduli spaces can be formulated by compactifying a 5+1D theory with (at least) $\mathcal{N} = (1, 0)$ supersymmetry on $\mathbf{T}^3$. The statement

3

of [17] was generalized in [22] (see also [15]) to the equivalence of the moduli space of compactified little-string-theories of NS5-branes at ADE singularities and instantons on $\mathbf{T}^4$. A further generalization of this statement was studied in [23]. There, it was argued that the moduli space of little-string-theories compactified on $\mathbf{T}^4$ with twisted boundary conditions is equivalent to the moduli space of instantons on a noncommutative $\mathbf{T}^4$. This also generalizes the result of [24] who showed that the moduli space of mass deformed 2+1D $\mathcal{N}=8$ gauge theories is equivalent to the moduli space of instantons on a noncommutative $\mathbf{S}^1 \times \mathbf{R}^3$.

These results give a realization of the hyper-Kähler moduli spaces of instantons on $\mathbf{T}^4$ and its deformation to noncommutative instantons. In fact, it suggests a natural definition of instanton moduli spaces of other simply-laced groups, like $E_{6,7,8}$, on a noncommutative $\mathbf{T}^4$ (even though the SYM theory is not defined)!

In this paper we will realize another class of hyper-Kähler moduli-spaces, namely, instantons on K_3 and, in particular, the deformation to instantons on a noncommutative K_3. We are going to argue that the moduli space of heterotic little-string theories [25] compactified on $\mathbf{T}^3$ with twisted boundary conditions is equivalent to the moduli space of instantons on a noncommutative K_3. (This result was also suggested in [26].) In particular, from special limits of this result one can obtain the moduli space of certain 3+1D $Sp(k)$ gauge theories with matter in the fundamental representation.

The relation between moduli-spaces of 5+1D theories that are compactified on $\mathbf{T}^3$ and moduli spaces of instantons on elliptically fibered spaces (K_3 or $\mathbf{T}^4$) makes it clear that the latter should have a spectral-curve type construction. To see this, we can compactify on $\mathbf{T}^2 \times \mathbf{S}^1$ and take the limit that the radius of $\mathbf{S}^1$ is large. In this case, we can first compactify on $\mathbf{T}^2$ and then compactify the low-energy limit of the 3+1D theory on the large $\mathbf{S}^1$. The low-energy limit of the 3+1D theory is constructed from a Seiberg-Witten curve as in [27] and the 2-step compactification makes it clear that the 2+1D moduli spaces are fibrations over the moduli space of Seiberg-Witten curves (of some genus g), with the fiber being the Jacobian (the $\mathbf{T}^{2g}$ space of $U(1)$ gauge field configurations with zero field-strength) of the curve [21]. This is precisely the spectral-curve construction – the spectral-curve being identified with the Seiberg-Witten curve.

The paper is organized as follows. In section (2) we briefly review the spectral-curve

constructions of holomorphic vector bundles on $\mathbf{T}^4$ and K_3. In section (3) we extend the spectral-curve construction of instanton moduli-spaces to the noncommutative $\mathbf{T}^4$. We show explicitly how the spectral curves are constructed in a twisted $\mathbf{T}^4$. In section (4) we extend the construction to a noncommutative K_3. We demonstrate explicitly, to first order in the noncommutativity, the spectral curves. We also discuss the instanton equations on a generic curved manifold, to first order in the noncommutativity. In section (5) we discuss the connection between noncommutative instanton moduli spaces on K_3 and moduli spaces of compactified heterotic little-string theories. In section (6) we generalize to the theories of heterotic NS5-branes at A_k singularities. In section (7) we describe the relation between Seiberg-Witten curves of the little-string theories compactified on $\mathbf{T}^2$ with twists, and the spectral-curve constructions of noncommutative instanton moduli spaces. We also describe the derivation of the spectral curves of K_3 from T-duality.

2   Spectral curve constructions

Holomorphic vector bundles on an eliptically fibered complex manifolds $\mathcal{M}$ with a section can be described with **spectral-curves**. We are interested in instanton moduli spaces and therefore we will take $\mathcal{M}$ to be a (real) 4-dimensional manifold. We will follow [10, 11].

Let us denote the base of the fibration by B and let us denote the $\mathbf{T}^2$ fiber over a generic point $p \in B$ by F_p. Let $\pi : \mathcal{M} \to B$ be the projection. The idea is to first study the restriction of the bundle to each F_p. A rank-q holomorphic vector bundle, V, on $\mathbf{T}^2$ with zero first Chern class, can be described naturally by q points on $\mathbf{T}^2$. These points then span a holomorphic curve $\Sigma \subset \mathcal{M}$ which is, in general, a q-fold cover of B with a certain number of branch points $p_1, \ldots, p_r \in B$. Let the multiplicity of the jth branch-point be n_j. Given Σ we can construct the restriction of the vector bundle to each F_p. This describes a vector bundle U that is locally a product of q line-bundles $\prod_{i=1}^{q} \mathcal{L}_i$ (over the local patch of $\mathcal{M}$). The structure group is locally a subgroup of $\mathbf{C}^q$. When we go around a branch point, p_j, we have to permute $n_j \leq q$ of the $\mathcal{L}_i$'s and this is how the nonabelian factor which is an element of S_{n_j} (the permutation group) enters into the structure group. To describe V, we still have the freedom to multiply by the pullback $\pi^*\mathcal{L}$ of a line-bundle $\mathcal{L}$ over the base B.

The instanton number, $c_2(V)$, is related to the homology class $[\Sigma]$ of the curve as [10, 11]:

$$[\Sigma] = q[B] + c_2(V)[F], \tag{1}$$

where $[B], [F]$ are the homology classes of B and F, respectively.

For example, for $\mathcal{M} = \mathbf{T}^4$ we can relate $c_2(V)$ to the genus of Σ as:

$$2g - 2 = 2qc_2(V),$$

while for $\mathcal{M} = K_3$ we have:

$$2g - 2 = 2q(c_2(V) - q).$$

In general, the spectral curve could have several components and we should replace $(2g - 2)$ with the sum over the distinct components.

3 The moduli space of instantons on $\mathbf{T}^4$

Let us demonstrate the spectral curves for $\mathbf{T}^4$. Let us, for simplicity, take $\mathbf{T}^4$ of the form $\mathbf{T}^2 \times \mathbf{T}^2$ and call the first factor, the "base" B, and the second factor, the "fiber" F. We take z to be a local coordinate on the base, such that

$$z \sim z + 1 \sim z + \sigma$$

We also take w to be a local coordinate on the fiber with:

$$w \sim w + 1 \sim w + \tau. \tag{2}$$

Let us take the rank to be q and $c_2 = k$. We restrict ourselves to vector-bundles with $c_1 = 0$. The restriction of the vector-bundle to any given fiber is a holomorphic vector bundle on $\mathbf{T}^2$ with $c_1 = 0$. Let p_0 denote the origin of F. Any line bundle with $c_1 = 0$ on F is equivalent to a divisor of the form $p - p_0$ with $p \in F$. The vector-bundle of rank-q can be reduced to a sum of q line-bundles and different orderings are equivalent. Thus, the vector-bundle can be described by q points in F. The position of the points is given locally by q maps:

$$w_a(z) : B \mapsto F, \qquad a = 1 \ldots q.$$

The boundary conditions are:

$$w_a(z + 1) = w_{\rho(a)}(z) + n_a + m_a \tau, \ (a = 1 \ldots q), \ \rho \in S_q, \ n_a, m_a \in \mathbf{Z},$$

and similarly for $z + \sigma$.

3.1   Small fiber

One can rederive the spectral curves for instantons on $\mathbf{T}^4$ by solving the instanton equations directly, in the limit that the area of the fiber F is zero. We can take an instanton solution described by the $U(q)$ gauge fields A_z, A_w and the complex conjugate $\overline{A_{\bar{z}}}$, $\overline{A_{\bar{w}}}$. The instanton equations imply:

$$0 = F_{zw} \equiv \partial_z A_w - \partial_w A_z - i[A_z, A_w].$$

If the fiber is very small, we can assume that A_w and A_z are independent of w. We can then also make a gauge transformation that ensures that A_w is in a $U(1)^q \subset U(q)$ Cartan subgroup. In the projection of the equation $F_{zw} = 0$ on this Cartan subgroup, the commutator term drops and we find that $\partial_z A_w = 0$ and $\overline{A_{\bar{w}}}$ is holomorphic. $\overline{A_{\bar{w}}}$ is a collection of q periodic variables and can be naturally identified with q points in the dual of the fiber. The complex structure of the dual of the fiber is still τ. So $\overline{A_{\bar{w}}}$ defines a holomorphic q-valued map from B to the dual of F and the graph, Σ, of this map is the spectral-curve. The dual of F has the same complex structure, τ. So, as far as complex structure goes, the instanton solution defines a curve Σ inside $\mathbf{T}^4$. Locally we denote the q points by:

$$w_a(z), \ (a = 1 \ldots q).$$

The boundary conditions are that:

$$w_a(z) = w_{\rho(a)}(z+1) = w_{\rho'(a)}(z+\sigma),$$

for some permutations $\rho, \rho' \in S_q$. If we set $A_z = 0$, we can also calculate the instanton number as:

$$\int \mathrm{tr}\{F \wedge F\} = \int |F_{zw}|^2 = \sum_{a=1}^{q} \int |\partial_z w_a|^2 dz = \int_\Sigma d^2 w = k.$$

Here k is the number of times the curve Σ intersects the section $w = 0$.

3.2   Noncommutativity turned on

We now turn on a constant bivector (contravariant 2-tensor) θ that makes $\mathbf{T}^4$ noncommutative and see how it affects the equations. In general, θ has 6 (real) independent components, but we will assume that only two of them are turned on. In the complex coordinates z, w,

we will take it to be

$$\theta \equiv \theta^{zw} = -\theta^{wz}, \overline{\theta} \equiv \overline{\theta^{zw}} = (\theta^{zw})^*.$$

Instantons on a noncommutative $\mathbf{T}^4$ have been studied in [28] from an algebraic approach. Noncommutative gauge theory on $\mathbf{T}^d$ has also been studied in [29]-[40]. Below, we will describe a more direct physical approach.

The instanton equations, in the noncommutative case, imply:

$$\overline{\lambda}I = F_{\overline{zw}} \equiv \partial_{\overline{z}}A_{\overline{w}} - \partial_{\overline{w}}A_{\overline{z}} - iA_{\overline{z}} \star A_{\overline{w}} + iA_{\overline{w}} \star A_{\overline{z}}. \tag{3}$$

Here λ is an unknown constant that, as we shall soon see, is determined by θ. θ also enters into the definition of the $\star$-product on the RHS. Next, we will assume that $A_{\overline{z}} = 0$ and obtain $\overline{\lambda}I = \partial_{\overline{z}}A_{\overline{w}}$. (We used $\partial_w A_{\overline{z}} = \partial_w A_{\overline{w}} = 0$.) Let us, to start with, take the gauge group to be $U(1)$. The last equation implies that $A_{\overline{w}}$ is given in terms of a holomorphic function $\phi(z)$ as:

$$A_{\overline{w}} = \phi(z) + \overline{\lambda}\overline{z}.$$

The boundary conditions are:

$$A_{\overline{w}}(z, \overline{z}) = A_{\overline{w}}(z + 1, \overline{z} + 1) = A_{\overline{w}}(z + \sigma, \overline{z} + \overline{\sigma}).$$

Let us first assume, naively, that $\frac{\tau_2}{\pi i}A_{\overline{w}}$ takes its values in a $\mathbf{T}^2$ with the same complex structure τ, in the notation of (2). We will see shortly that this assumption is not quite right, but let us proceed with it for the time being. Let us denote by u a complex coordinate on this $\mathbf{T}^2$ with $u \sim u + 1 \sim u + \tau$. Together, (z, u) would define a $\mathbf{T}^4$ that is identical in complex structure to the original $\mathbf{T}^4$. $A_{\overline{w}}$ would define a curve $\Sigma \subset \mathbf{T}^4$ by the equation:

$$\Sigma = \left\{ (z, u) : 0 = u - \frac{\tau_2}{\pi i}A_{\overline{w}}(z, \overline{z}) \right\}.$$

We can change the complex structure on $\mathbf{T}^4$ so that locally holomorphic functions $\psi(z, \overline{z}, u, \overline{u})$ will be defined to satisfy:

$$\partial_{\overline{u}}\psi = (\partial_{\overline{z}} + \frac{\tau_2}{\pi i}\overline{\lambda}\partial_u)\psi = 0.$$

We can do this by defining:

$$z_1 \equiv z, \qquad u_1 = u + \frac{\tau_2}{\pi i}\overline{\lambda}\overline{z}. \tag{4}$$

$$\partial_{\overline{u}_1} \equiv \partial_{\overline{u}}, \qquad \partial_{\overline{z}_1} \equiv \partial_{\overline{z}} + \frac{\tau_2}{\pi i}\overline{\lambda}\partial_u$$

In this complex structure, the curve Σ that is defined by the equation:

$$0 = u - \frac{\tau_2}{\pi i} A_{\overline{w}}(z, \overline{z}) = u - \frac{\tau_2}{\pi i}(\phi(z) + \overline{\lambda}\overline{z}),$$

is holomorphic. Naively, we now face a puzzle! If $u - \frac{\tau_2}{\pi i}(\phi(z) + \overline{\lambda}\overline{z})$ indeed defines a curve Σ, its cohomology class should be integral and given by an equation like (1):

$$[\Sigma] = q[B] + k[F].$$

But this implies that $[\Sigma]$ is an analytic curve in the complex structure defined by (u, z) and it cannot be also analytic in the other complex structure defined by (u_1, z_1) for $\lambda \neq 0$. In particular, if Σ is analytic in the complex structure defined by u_1, z_1, we can calculate:

$$\int_{\Sigma} du \wedge dz = \int_{\Sigma} du_1 \wedge dz_1 + \frac{\tau_2 \overline{\lambda}}{\pi i} \int_{\Sigma} dz \wedge d\overline{z} = \frac{2\tau_2 \overline{\lambda}}{\pi} A \neq 0.$$

where $A = \int_{\Sigma} \frac{dz \wedge d\overline{z}}{2i}$ is the area of the base B.

The resolution of the puzzle has to do with the noncommutative nature of the gauge group. What are the periodicity conditions imposed on $A_{\overline{w}}$? In the commutative case we had

$$\frac{\tau_2}{\pi} A_{\overline{w}} \sim \frac{\tau_2}{\pi} A_{\overline{w}} + i(n + m\tau), \qquad n, m \in \mathbf{Z}. \tag{5}$$

where $\tau = \tau_1 + i\tau_2$. Where did this periodicity condition come from? There are large gauge transformations $\Lambda(w, \overline{w})$ on the fiber that preserve the condition that $A_{\overline{w}}$ is a constant but shift it as follows. For:

$$\Lambda = e^{\frac{\pi}{\tau_2}[m(\tau \overline{w} - \overline{\tau} w) - n(w - \overline{w})]}, \tag{6}$$

that is single-valued on $\mathbf{T}^2$, we have:

$$A_{\overline{w}} \sim \Lambda^{-1} A_{\overline{w}} \Lambda + i\Lambda^{-1}\partial_{\overline{w}}\Lambda = A_{\overline{w}} + \frac{i\pi}{\tau_2}(n + m\tau)$$

and this implies the periodicity (5). In the noncommutative case we should calculate instead:

$$A_{\overline{w}}(z, \overline{z}) \sim \Lambda^{-1} \star A_{\overline{w}}(z, \overline{z}) \star \Lambda + i\Lambda^{-1} \star \partial_{\overline{w}}\Lambda = A_{\overline{w}}\left(z - \frac{2\pi i\theta}{\tau_2}(n + m\overline{\tau}), \overline{z} + \frac{2\pi i\overline{\theta}}{\tau_2}(n + m\tau)\right) + \frac{\pi i}{\tau_2}(n + m\tau).$$

This means that it was incorrect to interpret the curve Σ as a curve inside the original $\mathbf{T}^4$. Rather, we should define a modified $\widetilde{\mathbf{T}^4}$ with the identifications:

$$(u, z) \sim (u, z + n' + m'\sigma) \sim (u + n + m\tau, z - \frac{2\pi i\theta}{\tau_2}(n + m\overline{\tau})).$$

9

The curve Σ that we defined by the equation:

$$0 = u - \frac{\tau_2}{\pi i} A_{\overline{w}}(z, \overline{z}) = u - \frac{\tau_2}{\pi i}(\phi(z) + \overline{\lambda}\overline{z}),$$

is uniquely defined inside $\widetilde{\mathbf{T}}^4$ and not the original $\mathbf{T}^4$! For a $U(q)$ gauge-group the derivation is similar. We take the gauge transformation (6) in a $U(1)$ subgroup.

Now we can determine the relation between λ and θ. If Σ is to be holomorphic in the complex structure defined by (u_1, z_1) as in (4) then:

$$0 = \int_{\Sigma} du_1 \wedge dz_1 = q \int_{B'} du_1 \wedge dz_1 + k \int_{F'} du_1 \wedge dz_1. \tag{7}$$

The base B' and fiber F' of $\widetilde{\mathbf{T}}^4$ are defined from the periodicity conditions by:

$$
\begin{aligned}
B' &= \left\{ (z = \alpha + \beta\sigma, u = 0) : 0 \leq \alpha, \beta \leq 1 \right\}, \\
F' &= \left\{ \left(z = -\frac{2\pi i\theta}{\tau_2}(\alpha + \beta\overline{\tau}), u = \alpha + \beta\tau\right) : 0 \leq \alpha, \beta \leq 1 \right\},
\end{aligned}
\tag{8}
$$

Then, from (7), and using:

$$
\begin{aligned}
\int_{B'} du_1 \wedge dz_1 &= \frac{2\overline{\lambda}\sigma_2\tau_2}{\pi}, \\
\int_{F'} du_1 \wedge dz_1 &= \int_{F'} du \wedge dz - \frac{\overline{\lambda}\tau_2}{i\pi} \int_{F'} dz \wedge d\overline{z} = -4\pi\theta - 8\pi\overline{\lambda}|\theta|^2.
\end{aligned}
$$

one finds:

$$\overline{\lambda} = \frac{2\pi^2 k\theta}{q\sigma_2\tau_2 - 4\pi^2 k|\theta|^2}. \tag{9}$$

This agrees with the formula in [28].

3.3   Irreducibility of the curves

The moduli spaces of commutative instantons have singularities. On $\mathbf{R}^4$ the singularities correspond to instantons of zero size. On $\mathbf{T}^4$, the singular instanton configurations correspond to reducible spectral curves. We have seen that to describe a commutative instanton we need to find a holomorphic curve in the class:

$$[\Sigma] = q[B] + k[F].$$

A reducible instanton configuration will be of the form $\Sigma_1 \cup \Sigma_2$ where:

$$[\Sigma]_1 = q_1[B] + k_1[F], \quad [\Sigma]_2 = q_2[B] + k_2[F], \qquad q = q_1 + q_2, \ k = k_1 + k_2. \qquad (10)$$

It can be thought of as an embedding of the instantons in a $U(q_1) \times U(q_2) \subset U(q)$ subgroup such that the $U(q_1)$ instantons have instanton number k_1. One of the advantages of turning on the noncommutativity is that the moduli spaces can become nonsingular. For $\mathbf{R}^4$ this has been shown in [4, 5].

In the case of $\mathbf{T}^4$ we can show that the noncommutative spectral-curves are always irreducible if the greatest common divisor, $gcd(k, q) = 1$. First note that we cannot split the k instantons in two by embedding k_1 instantons in $U(q_1)$ and the other k_2 in $U(q_2)$. This is because λ in (3) depends on k (see (9)) and unless $k_1/q_1 = k_2/q_2$ we will have to have different λ's for $U(q_1)$ and $U(q_2)$ which is prohibited. In general, suppose that a certain spectral curve is reducible as in (10). Both Σ_1 and Σ_2 would have to be holomorphic in the same complex structure as Σ. If $gcd(k, q) = 1$ this would imply that $[F]$ should be also analytic (in $H^{(1,1)}$) for that particular complex structure. But as we have seen, for generic θ this is not the case, so the curves are always irreducible for $gcd(k, q) = 1$.

The moduli-space itself is a $\mathbf{T}^{2kq}$-fibration (the dual of the Jacobian) over the moduli space of spectral-curves. Although the fibers always correspond to irreducible curves they can become degenerate when cycles shrink to zero size.

3.4   The spectral bundle

So far we have set $A_{\bar{z}} = 0$. Let us set $A_{\bar{z}} \neq 0$ but keep $A_{\bar{z}}$ a function of $(z, \bar{z})$ alone. The third instanton equation implies:

$$\rho I = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + \partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + iA_z \star A_{\bar{z}} - iA_{\bar{z}} \star A_z + iA_w \star A_{\bar{w}} - iA_{\bar{w}} \star A_w = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z.$$

Here we used the fact that $A_{\bar{w}}$ is independent of w. As in the commutative case, it can be seen that $\rho = 0$ (for $c_1 = 0$) and that $A_{\bar{w}}$ can be lifted to a $U(1)$ (commutative) flat connection over the spectral curve Σ.

The spectral bundle is described by a flat connection on the curve Σ, of genus g (which is generically $kq + 1$). It is therefore specified by specifying a point in the dual of the

11

Jacobian of the curve. The Jacobian is $\mathbf{T}^{2g}$ and a point on this $\mathbf{T}^{2g}$ corresponds to a map from $\pi_1(\Sigma) \to U(1)$, that describes the $U(1)$ Wilson lines around the 1-cycles of Σ. The holomorphic structure of the Jacobian varies when the curve is varied. The moduli space of instantons is thus, locally, a $\mathbf{T}^{2g}$ fibration over the moduli space of Σ's.

Let us also note that the embedding $\rho : \Sigma \to \widetilde{\mathbf{T}^4}$ induces a map $\rho^* : \pi_1(\Sigma) \to \pi_1(\widetilde{\mathbf{T}^4})$. Let us take $\tilde{\gamma}_1, \tilde{\gamma}_2$ to be generators of $\pi_1(F') \subset \pi_1(\widetilde{\mathbf{T}^4})$, i.e. corresponding to 1-cycles on the fiber, F'. Let us take $\tilde{\gamma}_3, \tilde{\gamma}_4$ to be generators of $\pi_1(B') \subset \pi_1(\widetilde{\mathbf{T}^4})$, i.e. corresponding to 1-cycles on the base, B'. Then for $i = 1, 2$ $\rho^*(\gamma_i) = q\tilde{\gamma}_i$, where γ_i are generators of $\pi_1(\Sigma)$. For $i = 3, 4$ we have similarly, $\rho^*(\gamma_i) = k\tilde{\gamma}_i$. The $U(1)$ holonomies along the cycles corresponding to $\gamma_1, \ldots, \gamma_4$, generate a $\mathbf{T}^4$ subset of the Jacobian $\mathbf{T}^{2g}$. This $\mathbf{T}^4$ is actually the dual, $\mathbf{T}^{4\vee}$, of the original $\mathbf{T}^4$. It is not hard to see that as the curve Σ varies the dual of the Jacobian is locally of the form $\mathbf{T}^{4\vee} \times \mathbf{T}^{2g-4}$. Here $\mathbf{T}^{4\vee}$ is the dual of the original $\mathbf{T}^4$. (In subsection (3.6) we will see an explicit example.) The complex structure of $\mathbf{T}^{2g-4}$ varies when Σ varies but the complex structure of the $\mathbf{T}^{4\vee}$ remains fixed. Globally, the moduli space is a $\mathbf{T}^{4\vee}$ fiber-bundle. The fiber is this fixed $\mathbf{T}^{4\vee}$ that corresponds to the $U(1)$ holonomies along $\gamma_1, \ldots, \gamma_4$. The structure group of this fiber-bundle is $\mathbf{Z}_q^2 \otimes \mathbf{Z}_k^2$, because shifting the holonomy along γ_1, say, by $\frac{2\pi}{k}$, does not change the induced holonomy along $\tilde{\gamma}_1$.

3.5 Explicit formulas for the curves

We can write down explicit formulas for the curves using Θ-functions on $\mathbf{T}^4$. (See [41] for more details.) Let us choose real coordinates $x_1, \ldots, x_4$ such that $0 \le x_i \le 1$ and the periodic boundary conditions are given by $x_i \sim x_i + 1$. Let us denote by e_i the path from $x_i = 0$ to $x_i = 1$ keeping the other x_j's equal to zero. Consider the cohomology class:

$$\psi = m_1 dx_1 \wedge dx_3 + m_2 dx_2 \wedge dx_4.$$

where $k = m_1$ and $q = m_2$ are positive integers. We wish to find explicit formulas for holomorphic curves in a homology class $[\Sigma]$ that is Poincare dual to ψ.

A $\mathbf{T}^4$ has a $\mathbf{P}^1$-worth of complex structures. Given a particular complex structure, one can describe it by picking complex coordinates z_1, z_2 that are linear combinations of the x_i's. Of course, z_1, z_2 can be chosen up to a $GL(2, \mathbf{C})$ transformation. It can be shown (see [41])

that if there exists a holomorphic curve of class $[\Sigma]$ then z_1, z_2 can be chosen such that the 2×4 periods are of the form:

$$
\begin{aligned}
\int_{e_i} dz_\alpha &= \delta_{i\alpha} m_i, & i &= 1, 2, & \alpha &= 1, 2, \\
\int_{e_{i+2}} dz_\alpha &= Z_{i\alpha}, & i &= 1, 2 & \alpha &= 1, 2,
\end{aligned}
$$

and the 2×2 matrix Z is symmetric and its imaginary part, $\mathrm{Im}\, Z$, is a positive matrix.

Now define $m_1 m_2 = kq$ θ-functions as follows. For $0 \le l_1 < m_1$ and $0 \le l_2 < m_2$ we set:

$$
\Theta_{l_1 l_2}(z_1, z_2) \equiv \sum_{N_1, N_2 \in \mathbf{Z}} e^{\pi i \sum_{\alpha,\beta=1,2} Z_{\alpha\beta} N_\alpha (N_\beta + 2 m_\beta^{-1} l_\beta) + 2\pi i \sum_{\alpha=1,2} (N_\alpha + m_\alpha^{-1} l_\alpha) z_\alpha}.
$$

Let us look at an equation of the form:

$$
0 = \sum_{l_1=0}^{m_1-1} \sum_{l_2=0}^{m_2-1} A_{l_1 l_2} \Theta_{l_1 l_2}(z_1, z_2) \tag{11}
$$

It depends on the $m_1 m_2 = kq$ complex parameters $A_{l_1 l_2}$. It can be shown to describe a curve of class $[\Sigma]$. The Θ-functions are linearly independent. Since the overall factor is unimportant, we see that the moduli space of such curves is equivalent to $\mathbf{P}^{kq-1}$. The moduli space of curves of class $[\Sigma]$ is of dimension $kq + 1$. The missing parameters can be described as translations of the curve as a whole. The translations are described by two parameters, one for $z_1 \to z_1 + \epsilon$ and the other for $z_2 \to z_2 + \epsilon$.

3.6 Examples: Instanton number $k = 1, 2$

As a simple example, let us take $q = 1$ and $k = 1$. The spectral curves are of genus $g = 2$ and there is only one such curve inside $\widetilde{\mathbf{T}^4}$, up to translations. Thus the moduli space of spectral curves is $\widetilde{\mathbf{T}^4}$. The dual of the Jacobian of the curve is a fixed $\mathbf{T}^4{}_J$. According to the discussion at the end of subsection (3.4), this $\mathbf{T}^4{}_J$ is the dual $\widetilde{\mathbf{T}^4}^\vee$ and the overall moduli-space is of the form $\widetilde{\mathbf{T}^4} \times \widetilde{\mathbf{T}^4}^\vee$.

As a second example, let us take $q = 1$ and $k = 2$, that is, two $U(1)$ instantons. The commutative moduli space would be formally the space of point-like instantons, $(\mathbf{T}^4)^2 / \mathbf{Z}_2$ multiplied by an extra $\mathbf{T}^4$ factor corresponding to the overall $U(1)$ holonomies. The overall $\mathbf{T}^4$ factor corresponds to the 4 overall degrees of freedom of translating the spectral curve as a whole.

13

From the spectral-curve construction, we obtain spectral curves that are of genus $kq+1 = 3$. Their Jacobian is a varying $\mathbf{T}^6$. As we have seen in subsection (3.4), we can separate the $\mathbf{T}^6$ into a locally fixed $\mathbf{T}^4$ and a varying $\mathbf{T}^2$. The moduli-space of spectral curves is $\mathbf{P}^1$. and we therefore obtain a $\mathbf{T}^2$ fibration over $\mathbf{P}^1$ which must correspond to a K_3. If we add the extra $\mathbf{T}^{4\vee}$ factor that we separated from the dual of the Jacobian we get a moduli space that is a $\mathbf{T}^{4\vee}$ fiber-bundle over K_3, with the structure group being $\mathbf{Z}_2$ (corresponding to shifts by half lattice-vectors in the fiber $\mathbf{T}^4$).

We can be more precise and calculate the number of singular fibers over this $\mathbf{P}^1$. From (11) we see that the curves inside $\mathbf{T}^4$ are given by an equation of the form $0 = \Theta_{00} - \lambda\Theta_{10}$, where Θ_{00} and Θ_{10} are two sections of the same line-bundle over $\mathbf{T}^4$ and $\lambda \in \mathbf{P}^1$ is a coordinate on the moduli space of spectral-curves. For a generic point in $\mathbf{T}^4$, the values of $[\Theta_{00}, \Theta_{10}]$ define a unique point in $\mathbf{P}^1$. However, there are 4 points inside $\mathbf{T}^4$ where $\Theta_{00} = \Theta_{10} = 0$. (The number 4 is obtained by calculating the intersection number of the two curves $\Theta_{00} = 0$ and $\Theta_{10} = 0$.) These points do not define a unique point in $\mathbf{P}^1$, but in the manifold $\mathbf{T}^4$ with 4 points blown up, every point corresponds to a unique point on $\mathbf{P}^1$. Thus, generically, this manifold is a fibration of the genus-3 spectral-curves over the base $\mathbf{P}^1$. The Euler number of $\mathbf{T}^4$ with 4 blown up points is $\chi = 4$. On the other hand, the Euler number of a genus-3 Riemann surface fibered over $\mathbf{P}^1$ is $\chi(\mathbf{P}^1) \times \chi(\Sigma_3) = -8$. The mismatch is accounted for by noting that there must be d points on the base $\mathbf{P}^1$ where a cycle of the spectral-curve shrinks. This would bring the Euler number up to $\chi = -8 + d$ and for $d = 12$ we get a match.

For $[\lambda_1, \lambda_2] \in \mathbf{P}^1$, the curve:

$$\lambda_1 \Theta_{00}(z_1, z_2) + \lambda_2 \Theta_{10}(z_1, z_2) = 0,$$

is singular when:

$$\lambda_1 \partial_{z_i} \Theta_{00}(z_1, z_2) + \lambda_2 \partial_{z_i} \Theta_{10}(z_1, z_2) = 0, \qquad i = 1, 2.$$

Eliminating the λ_i's, this gives us two equations:

$$\Phi_i(z_1, z_2) \equiv \Theta_{10} \partial_{z_i} \Theta_{00} - \Theta_{00} \partial_{z_i} \Theta_{10} = 0, \qquad i = 1, 2.$$

Note that if Θ_{00} and Θ_{10} are sections of the same line-bundle $\mathcal{L}$, then Φ_i are sections of $\mathcal{L}^2$. The divisor of zeroes for each Φ_i is therefore $2[B] + 4[F]$. Thus the number of solutions to

14

$\Phi_1 = \Phi_2 = 0$ is 16. Out of these, 4 solutions correspond to $\Theta_{00} = \Theta_{10} = 0$. These are not singular points, since the derivatives are in general nonzero. Thus we are left with 12 singular points.

For $m_1 = 2$ and $m_2 = 1$ we have:

$$\Theta_{00}(z_1, z_2) = \sum_{N_1, N_2 \in \mathbf{Z}} e^{\pi i (Z_{11} N_1^2 + 2Z_{12} N_1 N_2 + Z_{22} N_2^2 + 2N_1 z_1 + 2N_2 z_2)},$$

$$\Theta_{10}(z_1, z_2) = \sum_{N_1, N_2 \in \mathbf{Z}} e^{\pi i (Z_{11} N_1^2 + 2Z_{12} N_1 N_2 + Z_{22} N_2^2 + 2N_1 z_1 + 2N_2 z_2 + Z_{11} N_1 + Z_{12} N_2 + z_1)},$$

The $\mathbf{T}^4$ is generated by the columns of the matrix:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2}Z_{11} & \frac{1}{2}Z_{21} \\ 0 & 1 & Z_{12} & Z_{22} \end{pmatrix}, \qquad Z_{21} = Z_{12}. \tag{12}$$

Now let us describe the Jacobian of the $g = 3$ curve. The Jacobian is a $\mathbf{T}^6$ and to describe it, we need to pick 3 holomorphic 1-forms, ψ_i $(i = 1, 2, 3)$ on Σ and Integrate them over the 6 1-cycle generators of $H_1(\Sigma)$. This will give us a basis for a lattice in $\mathbf{C}^3$. Let $\rho : \Sigma \subset \mathbf{T}^4$ be the embedding of the curve. We can take $\psi_i = \rho^* dz_i$ $(i = 1, 2)$. Let C'_a $(a = 1 \ldots 6)$ be a$a = 1 \ldots 4$) be a basis of 1-cycles of Σ and let C_a $(a = 1 \ldots 4)$ be a basis of 1-cycles of $\mathbf{T}^4$ corresponding to the columns of (12). We can choose C'_a such that:

$$\rho(C'_1) = C_1, \ \rho(C'_2) = \rho(C'_3) = C_2, \ \rho(C'_4) = C_3, \ \rho(C'_5) = \rho(C'_6) = C_4.$$

We can now choose $\psi_3 \in H^{(1,0)}(\Sigma)$ such that:

$$\int_{C'_1} \psi_3 = \int_{C'_2} \psi_3 = 0.$$

The period matrix for Σ (whose elements are the integrals $\int_{C'_a} \psi_i$) is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2}Z_{11} & \frac{1}{2}Z_{21} & \frac{1}{2}Z_{21} \\ 0 & 1 & 1 & Z_{12} & Z_{22} & Z_{22} \\ 0 & 0 & 1 & x & y & z \end{pmatrix}.$$

After a linear transformation this becomes:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2}Z_{11} & \frac{1}{2}Z_{12} & \frac{1}{2}Z_{12} \\ 0 & 1 & 0 & Z_{12} - x & Z_{22} - y & Z_{22} - z \\ 0 & 0 & 1 & x & y & z \end{pmatrix}.$$

15

The condition that Σ is analytic implies that the second 3×3 block of this matrix should be symmetric (see [41]). It follows that $x = \frac{1}{2}Z_{12}$ and $y = Z_{22} - z$. A holomorphic change of basis: $\psi_1' = \psi_1$, $\psi_2' = \psi_2 + \psi_3$ and $\psi_3' = \psi_3 - \psi_2$ and an $SL(6, \mathbf{Z})$ transformation renders the Jacobian in the form of a $\mathbf{T}^6$ generated by the columns of:

$$
\begin{pmatrix}
1 & 0 & \frac{1}{2}Z_{11} & Z_{12} & 0 & 0 \\
0 & 1 & Z_{12} & 2Z_{22} & 0 & 0 \\
0 & 0 & \frac{1}{2}Z_{12} & Z_{22} & w & 1
\end{pmatrix},
$$

Here $w = Z_{22} - 2z$. We actually need the dual of that $\mathbf{T}^6$. It is generated by:

$$
\begin{pmatrix}
1 + \frac{\operatorname{Re} Z_{12}\operatorname{Im} Z_{12} - \operatorname{Re} Z_{11}\operatorname{Im} Z_{22}}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
\frac{\operatorname{Im} Z_{12}\operatorname{Re} Z_{11} - \operatorname{Im} Z_{11}\operatorname{Re} Z_{12}}{2(\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21})}i \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
\frac{2(\operatorname{Re} Z_{22}\operatorname{Im} Z_{21} - \operatorname{Re} Z_{12}\operatorname{Im} Z_{22})}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
1 + \frac{\operatorname{Im} Z_{12}\operatorname{Re} Z_{21} - \operatorname{Im} Z_{11}\operatorname{Re} Z_{22}}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
\frac{\operatorname{Im} Z_{22}}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
-\frac{\operatorname{Im} Z_{12}}{2(\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21})}i \\
0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-\frac{\operatorname{Im} Z_{21}}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
\frac{\operatorname{Im} Z_{11}}{2(\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21})}i \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \quad
\begin{pmatrix}
-\frac{\operatorname{Re} Z_{22}\operatorname{Im} Z_{12} - \operatorname{Re} Z_{12}\operatorname{Im} Z_{22}}{\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21}}i \\
-\frac{1}{2} - \frac{\operatorname{Re} Z_{12}\operatorname{Im} Z_{12} - \operatorname{Re} Z_{22}\operatorname{Im} Z_{11}}{2(\operatorname{Im} Z_{11}\operatorname{Im} Z_{22} - \operatorname{Im} Z_{12}\operatorname{Im} Z_{21})}i \\
w
\end{pmatrix}.
$$

This describes the dual $(\mathbf{T}^4)^\vee$ fibered over the $\mathbf{T}^2$ with (variable) complex structure w. The fibration is described by a translation by half a lattice when going around one of the cycles of the $\mathbf{T}^2$.

4 Instantons on K_3

After having analyzed instantons on $\mathbf{T}^4$, we move on to noncommutative instantons on other complex manifolds where the metric is not flat. As in the case of $\mathbf{T}^4$, we wish to study the holomorphic structure of the moduli space of instantons.

As a first step, we have to discuss the form of the instanton equations on the noncommutative manifold. As in the $\mathbf{R}^4$ case, the instanton equations can be written down once one defines an associative $\star$-product on functions and on 1-forms. The constant bivector (antisymmetric contravariant tensor) θ^{ij} is now replaced with a covariantly constant bivector that we still denote by θ^{ij}. The construction of the $\star$-product can be found in [42, 43, 44]. We will only need the lowest order terms which we will review shortly.

We will also restrict ourselves to bivectors, θ^{ij}, such that the inverse matrix $(\theta^{-1})_{ij}$ is a sum of holomorphic and anti-holomorphic 2-forms. The only 4-dimensional compact manifolds with a covariantly constant holomorphic 2-form are $\mathbf{T}^4$ and K_3, so our next example is a K_3 surface. Furthermore, we will consider the special case of elliptically fibered K_3's with a section.

We will first review the constructions of commutative instantons on elliptically fibered K_3's and the construction of spectral curves. We will then discuss the instanton equations on a curved noncommutative space and we will calculate the corrections to the spectral curve construction for a K_3. Finally, we will compare with predictions from T-duality in section (7).

We will begin with some preliminaries from geometry.

4.1   The geometry of elliptically fibered K_3

Let us start by reviewing the geometry and the construction of holomorphic curves inside an elliptically fibered K_3. We can parameterize the K_3 as:

$$y^2 = x^3 - f_8(z)x - g_{12}(z),$$

where z is a coordinate on the base, $\mathbf{C}$, and (x, y) are coordinates on $\mathbf{C}^2$. The base is made into $\mathbf{P}^1$ by adding the point $z = \infty$ and defining the good coordinates near $z = \infty$ as follows:

$$w = \frac{1}{z}, \qquad \xi = \frac{x}{z^4}, \qquad \eta = \frac{y}{z^6}.$$

This means that there is a line-bundle $\mathcal{L} = \mathcal{O}(2)$ on $\mathbf{P}^1$ and x is a section of $\mathcal{L}^2$ and y is a section of $\mathcal{L}^4$.

We denote the homology class of the fiber by $[F]$ and the homology class of the section $x = y = \infty$ by $[B]$. We have the intersections:

$$[B] \cdot [B] = -2, \ [F] \cdot [F] = 0, \ [B] \cdot [F] = 1.$$

For any analytic curve $\Sigma \in K_3$, we have the adjunction formula:

$$2g(\Sigma) - 2 = \Sigma \cdot \Sigma.$$

An expression of the form [10]:

$$\sum_{k=0}^{n} a_k(z)x^k + y \sum_{k=0}^{m} b_k(z)x^k = 0,$$

defines an analytic curve. Here we take $a_k(z)$ to be a section of $\mathcal{L}_0 \otimes \mathcal{L}^{-2k}$ and $b_k(z)$ to be a section of $\mathcal{L}_0 \otimes \mathcal{L}^{-3-2k}$. We take $\mathcal{L}_0 = \mathcal{O}(q+l)$ with q being the largest of $3+2m$ and $2n$. If $2m+3 > 2n$, then $q = 2m+3$ and a_k is of degree $2m - 2k + l + 3$ and b_k is of degree $2m - 2k + l$. Let us count the number of solutions to $x = y = \infty$. The leading term behaves like $P_{2m-2n+l+3}(z)x^n + Q_l(z)yx^m = 0$, where $P_{2m-2n+l+3}$ and Q_l are polynomials of degrees $(2m - 2n + l + 3)$ and l respectively. If $x = \infty$ we need $y \sim x^{3/2}$. Near $x = y = \infty$, the good coordinate is $\rho \equiv x^{-1/2} \sim \frac{x}{y}$. For $2m+3 > 2n$ the equation becomes: $P_{2m-2n+l+3}(z)\rho^{2m+3-2n} + Q_l(z) = 0$. When we set $\rho = 0$ we have to solve $Q_l(z) = 0$. In general, $Q_l(z)$ will have l zeroes and this is the number of solutions of $x = y = \infty$. We equate this with $c_2 - 2q$ and find $c_2 = 2q + l$. If $2m + 3 < 2n$ we find in a similar fashion that $q = 2n$. Then a_k is of degree $2n - 2k + l$ and b_k is of degree $2n - 2k - 3 + l$. we have to solve $P_l(z)x^n + Q_{2n-2m-3+l}(z)yx^m = 0$ for $x = y = \infty$. We find that there are l solutions, as before, and therefore $c_2 = 2q + l$ again. The homology class of Σ is therefore:

$$[\Sigma] = q[B] + (2q+l)[F].$$

Its genus is given by:

$$2g(\Sigma) - 2 = \Sigma \cdot \Sigma = 2q(q+l).$$

When the curves are spectral curves that describe instantons of $U(q)$ at instanton number k, we identify $q = F \cdot \Sigma$ and $k \equiv c_2 = 2q + l$.

Later, it will be more convenient to change coordinates locally to w and a that we define as follows. We take w to be a local coordinate on the fiber and $a(z)$ to be a local coordinate on the base. Locally, and away from the singular fibers, we can fix a basis α, β of $H^1(F)$, the 1-cycles of the $\mathbf{T}^2$ fiber. We take w to be a coordinate on the fiber, with the identifications:

$$w \sim w + n + m\tau(z), \qquad n, m \in \mathbf{Z}$$

where $\tau(z)$ is the complex structure of the fiber (defined locally up to $SL(2, \mathbf{Z})$). w is chosen so that the integrations on the fiber give:

$$\oint_{\alpha} dw = 1, \qquad \oint_{\beta} dw = \tau(z).$$

18

We then define $a(z)$ locally, as in the Seiberg-Witten theory, by the formula:

$$\frac{da}{dz} = \oint_\alpha \frac{dx}{y}, \tag{13}$$

We will also need the integral on the other cycle, $a_D(z)$:

$$\frac{da_D}{dz} = \oint_\beta \frac{dx}{y}. \tag{14}$$

We now write the complex structure of the fiber as $\tau(a) = \tau_1 + i\tau_2$. The metric is given by:

$$ds^2 = \tau_2 da d\bar{a} + \frac{\rho}{\tau_2} \left| dw + i\frac{\tau'}{2\tau_2}(w - \overline{w})da \right|^2 .$$

Here ρ is the area of the fiber and we set it to $\rho = 1$, for simplicity.

4.2   Small fiber

As in the case of $\mathbf{T}^4$, we can solve the instanton equations directly in the limit that the area of the fiber shrinks to zero. In this subsection we will review the commutative $U(q)$ theories [10] and in the next subsection we will discuss the noncommutative case. The instanton equation implies:

$$0 = F_{\overline{z}\overline{w}} \equiv \partial_{\overline{z}}A_{\overline{w}} - \partial_{\overline{w}}A_{\overline{z}} - i[A_{\overline{z}}, A_{\overline{w}}],$$

Now we assume that the fiber $\mathbf{T}^2$ is small and therefore $A_{\overline{w}}$ is independent of w and $\overline{w}$. However, we should not conclude that $A_{\overline{z}}$ is independent of w because it is not periodic in w. Rather, under:

$$w \rightarrow w + \tau(z),$$

we have:

$$A_{\overline{z}} \rightarrow A_{\overline{z}} - \overline{\tau}'(\overline{z})A_{\overline{w}}.$$

Therefore:

$$A_{\overline{z}} + \frac{\overline{\tau}'(\overline{z})}{\tau(z) - \overline{\tau}(\overline{z})}(w - \overline{w})A_{\overline{w}}, \tag{15}$$

is periodic in w and we can assume that it is independent of w.

$$\partial_{\overline{z}}A_{\overline{w}} - \partial_{\overline{w}}A_{\overline{z}} = \partial_{\overline{z}}A_{\overline{w}} - \frac{\overline{\tau}'(\overline{z})}{\tau(z) - \overline{\tau}(\overline{z})}A_{\overline{w}} = \frac{1}{\tau_2}\partial_{\overline{z}}(\tau_2 A_{\overline{w}}).$$

19

We can choose $A_{\overline{w}}$ to be a diagonal $q \times q$ matrix. The instanton equation then becomes:

$$\partial_{\overline{z}}\left(\tau_2 A_{\overline{w}}\right) = 0.$$

$$A_{\overline{w}} = \frac{\pi}{\tau_2}\phi(z), \tag{16}$$

where $\phi(z)$ is analytic in z. The Wilson lines along α, β are related to $A_{\overline{w}}$ as:

$$\int_\alpha A = 2\mathrm{Im}\, A_{\overline{w}}, \qquad \int_\beta A = 2\mathrm{Im}\left(\overline{\tau} A_{\overline{w}}\right).$$

$\tau_2 A_{\overline{w}}$ has the following periodicity:

$$\frac{\tau_2}{\pi}A_{\overline{w}} \sim \frac{\tau_2}{\pi}A_{\overline{w}} + n + m\tau(z), \qquad n, m \in \mathbf{Z}.$$

Locally, $\phi(z)$ defines q holomorphic functions from B with values in the fiber and they fit together to describe a holomorphic curve $\Sigma \subset K_3$.

4.3 Instanton equations on a noncommutative manifold

To define the instanton equations on a curved manifold we need to modify the $\star$-product. We assume that θ^{ij} is covariantly constant. (Throughout this section, we will use small letters $a, b, c, d, e, \ldots, i, j, \ldots$ for tensor indices.) The $\star$-product is an associative product on functions and tensors that to lowest order looks like:

$$T_{a_1\ldots a_p} \star S_{b_1\ldots b_q} \;=\; T_{a_1\ldots a_p}S_{b_1\ldots b_q} + i\theta^{cd}\nabla_c T_{a_1\ldots a_p}\nabla_d S_{b_1\ldots b_q} + O(\theta)^2,$$

Here ∇ is the covariant derivative:

$$\nabla_c T_{a_1\ldots a_p} \;\equiv\; \partial_c T_{a_1\ldots a_p} - \Gamma^b_{ca_1} T_{ba_2\ldots a_p} - \cdots - \Gamma^b_{ca_p} T_{a_1\ldots a_{p-1}b}.$$

where Γ^c_{ab} is the Christofel symbol. Associativity is required to determine the higher order corrections. The complete expression can be found in [42, 43, 45, 44].

On a curved manifold, the covariant derivative is not a derivation for the $\star$-product, i.e.

$$\nabla_c(T_{a_1\ldots a_p} \star S_{b_1\ldots b_q}) \neq \nabla_c T_{a_1\ldots a_p} \star S_{b_1\ldots b_q} + T_{a_1\ldots a_p} \star \nabla_c S_{b_1\ldots b_q}.$$

For example, it is easy to check that for a scalar and a vector:

$$\nabla_b(\phi \star A_a) - \nabla_b\phi \star A_a - \phi \star \nabla_b A_a \;=\; i\theta^{cd}\nabla_c\phi(\nabla_b\nabla_d - \nabla_d\nabla_b)A_a + O(\theta)^2$$
$$=\; i\theta^{cd}\nabla_c\phi R_{bda}{}^e A_e + O(\theta)^2.$$

20

Here $R_{abc}{}^d$ is the curvature tensor.

Now suppose we have a gauge field A_a. We can attempt to define the field-strength as:

$$F_{ab} = \nabla_a A_b - \nabla_b A_a - i A_a \star A_b + i A_b \star A_a.$$

The instanton equations will then be:

$$F_{ab} - \sqrt{g}\epsilon_{abcd} F^{cd} = \lambda'(\theta^{-1})_{ab},$$

where λ' is a constant. However, these equations will not be invariant under a gauge trans-
formation:

$$A_a \rightarrow i\Lambda^{-1} \star \nabla_a \Lambda + \Lambda^{-1} \star A_a \star \Lambda.$$

To first order we calculate:

$$F_{ab} = \nabla_a A_b - \nabla_b A_a - i A_a A_b + i A_b A_a + \theta^{kl}\nabla_k A_a \nabla_l A_b - \theta^{kl}\nabla_k A_b \nabla_l A_a + O(\theta)^2$$

For a $U(1)$ gauge field we then find:

$$F'_{ab} - \Lambda^{-1} \star F_{ab} \star \Lambda = \theta^{kl} R_{abl}{}^m[(\Lambda^{-1}\nabla_m\Lambda)(\Lambda^{-1}\nabla_k\Lambda) - 2iA_m\Lambda^{-1}\nabla_k\Lambda] + O(\theta)^2. \quad (17)$$

(We have used the identity $R_{alb}{}^m - R_{bla}{}^m = R_{abl}{}^m$.) In general, this term is nonzero. Note
that for a Hyper-Kähler 4-manifold, $R_{abl}{}^m$ is self-dual in the a, b indices. The anti-self-dual
part of the field-strength is therefore gauge invariant for Hyper-Kähler manifold.

4.4   Spectral curves for a noncommutative K_3

We now specialize to K_3. The noncommutativity is specified by a symplectic form that is
a bivector (an antisymmetric contravariant tensor of rank 2) θ^{ij}. We will denote by $(\theta^{-1})_{ij}$
the inverse 2-form such that $\theta^{ij}(\theta^{-1})_{jk} = \delta^i_k$. We also assume that θ^{-1} is a sum of a $(2,0)$
and $(0,2)$ forms. Let ω be a covariantly constant $(2,0)$ form on K_3. In terms of the local
holomorphic coordinates a and w, defined near (13), we have locally:

$$\omega = da \wedge dw = dz \wedge \frac{dx}{y}.$$

We take the $(2,0)$ part of θ to be proportional to ω so that:

$$\theta^{-1} = \vartheta^{-1} da \wedge dw + \overline{\vartheta}^{-1} \overline{a} \wedge d\overline{w}.$$

21

Here ϑ is a proportionality constant and $\overline{\vartheta}$ is its complex conjugate.

We now shrink the fiber to zero and take the metric to be:

$$ds^2 = \tau_2 dad\overline{a} + \frac{\rho}{\tau_2}\left|dw + i\frac{\tau'}{2\tau_2}(w - \overline{w})da\right|^2.$$

Here:

$$\tau_2 \equiv \frac{i}{2}(\overline{\tau}(\overline{a}) - \tau(a)), \qquad \tau' \equiv \tau'(a),$$

and $\rho \to 0$ is the area of the fiber. As in (15), the boundary conditions on A_a imply, for a small fiber, that:

$$A_a(w, \overline{w}) = \Xi(a, \overline{a}) + \frac{i\tau'}{2\tau_2}(w - \overline{w})A_w(a, \overline{a}).$$

We will set $\Xi(a, \overline{a}) = 0$.

Because K_3 is Hyper-Kähler, the curvature piece in (17) does not contribute to the instanton equations. The holomorphic part of the instanton equations becomes:

$$
\begin{aligned}
\lambda I &= \partial_a A_w - \partial_w A_a - iA_a \star A_w + iA_w \star A_a \\
&= \partial_a A_w - \frac{i\tau'}{2\tau_2}A_w - 2\theta^{aw}(\nabla_a A_w)(\nabla_w A_a) + 2\theta^{aw}(\nabla_w A_w)(\nabla_a A_a) \\
&\quad - 2\theta^{\overline{aw}}(\nabla_{\overline{a}}A_w)(\nabla_{\overline{w}}A_a) + 2\theta^{\overline{aw}}(\nabla_{\overline{w}}A_w)(\nabla_{\overline{a}}A_a)
\end{aligned}
\tag{18}
$$

We define:

$$\phi(a, \overline{a}) \equiv -\frac{i\tau_2 A_w}{\pi}.$$

Before we continue with the instanton equation, let us calculate the periodicity condition because of gauge invariance. We pick a gauge transformation:

$$\Lambda = e^{2\pi i m \frac{w - \overline{w}}{\tau - \overline{\tau}} - 2\pi i n \frac{\tau\overline{w} - \overline{\tau}w}{\tau - \overline{\tau}}}$$

and calculate the change under:

$$A_w \to i\Lambda^{-1} \star \partial_w \Lambda + \Lambda^{-1} \star A_w \star \Lambda.$$

We find:

$$
\begin{aligned}
\phi \to \phi &+ m + n\overline{\tau} + \frac{\pi\overline{\vartheta}}{2\tau_2^2}(m + n\tau)^2\overline{\tau}' \\
&- \frac{2\pi i\vartheta}{\tau_2}(m + n\overline{\tau})\partial_a\phi + \frac{2\pi i\overline{\vartheta}}{\tau_2}(m + n\tau)\partial_{\overline{a}}\phi + \frac{\pi\overline{\vartheta}}{\tau_2^2}(m + n\tau)\overline{\tau}'\phi + O(\vartheta)^2.
\end{aligned}
$$

22

These equations can be interpreted as follows. The periodicity equation for ϕ can be summarized in the relation:

$$(u, a + \delta a) \sim (u + m + n\tau + \delta u, a), \tag{19}$$

where:

$$\delta a \equiv -\frac{2\pi i \vartheta}{\tau_2}(m + n\overline{\tau}) + O(\vartheta)^2,$$

$$\delta u \equiv \frac{\pi \vartheta}{2\tau_2^2}(m + n\overline{\tau})^2 \tau' + \frac{\pi \vartheta}{\tau_2^2}(m + n\overline{\tau})\tau' u + O(\vartheta)^2,$$

This is chosen so that a curve given by the equation $\overline{u} = \phi(a, \overline{a})$ will be mapped by $\sim$ to the curve given by:

$$\begin{aligned}
\overline{u} = {} & \phi + m + n\overline{\tau} + \frac{\pi\overline{\vartheta}}{2\tau_2^2}(m + n\tau)^2 \overline{\tau}' \\
& -\frac{2\pi i\vartheta}{\tau_2}(m + n\overline{\tau})\partial_a\phi + \frac{2\pi i\overline{\vartheta}}{\tau_2}(m + n\tau)\partial_{\overline{a}}\phi + \frac{\pi\overline{\vartheta}}{\tau_2^2}(m + n\tau)\overline{\tau}'\phi + O(\vartheta)^2.
\end{aligned}$$

Now we can change coordinates to (a_0, u_0) with:

$$u_0 \equiv u + \frac{\pi\vartheta}{2\tau_2^2}(\overline{u}^2 - 2u\overline{u})\tau' + O(\vartheta)^2,$$

$$a_0 \equiv a - \frac{2\pi i\vartheta}{\tau_2}\overline{u} + O(\vartheta)^2,$$

In terms of the new coordinates (a_0, u_0), the identifications (19) become:

$$(a_0, u_0) \sim (a_0, u_0 + m + n\tau(a_0)).$$

Thus, the identification (19) defines, locally, a manifold that seems locally identical to the original K_3, but with a different metric!

It has an integral homology class, F', generated by the new fiber

$$a_0 = a - \frac{2\pi i\vartheta}{\tau_2}\overline{u} + O(\vartheta)^2 = \text{const.}$$

The base , B, is given by the equation $u_0 = 0$ (which is the same as $u = 0$).

Now we can proceed with (18). Expanding to $O(\theta)$, we find that (18) becomes:

$$\lambda = \partial_a A_w - \frac{i\tau'}{2\tau_2}A_w - \frac{i\overline{\vartheta}\tau'}{\tau_2}A_w\partial_{\overline{a}}A_w.$$

23

In terms of ϕ this reads:

$$0 = \partial_a\phi + \frac{i\lambda\tau_2}{\pi} - \frac{\pi i\overline{\vartheta}|\tau'|^2}{2\tau_2^3}\phi^2 + \frac{\pi\overline{\vartheta}\tau'}{\tau_2^2}\phi\partial_{\overline{a}}\phi + O(\vartheta)^2.$$

Now consider the curve given by the equation:

$$\Xi(u, a, \overline{a}) \equiv \overline{u} - \phi(a, \overline{a}) = 0.$$

Define the differential operators:

$$D_1 \equiv \partial_a - \frac{i\lambda\tau_2}{\pi}\partial_{\overline{u}} + \frac{\pi i\overline{\vartheta}|\tau'|^2}{2\tau_2^3}\overline{u}^2\partial_{\overline{u}} + \frac{\pi\overline{\vartheta}\tau'}{\tau_2^2}\overline{u}\partial_{\overline{a}},$$

$$D_2 \equiv \partial_u.$$

The instanton equation implies that, restricted to the curve $\Xi = 0$,

$$D_1\Xi|_{\Xi=0} = D_2\Xi = 0.$$

We can rewrite D_1 and D_2 in terms of u_0 and a_0 as:

$$
\begin{aligned}
D_1 &= -\left(1 + \frac{\pi\vartheta\tau'}{\tau_2^2}\overline{u}_0\right)\partial_{a_0} + \frac{\pi\overline{\vartheta}\tau'}{\tau_2^2}(u_0 - \overline{u}_0)\partial_{z_0} - \left(\frac{\pi i\vartheta}{2\tau_2^3}\tau'^2 + \frac{\pi\vartheta}{2\tau_2^2}\tau''\right)(\overline{u}_0^2 - 2u_0\overline{u}_0)\partial_{u_0} \\
&\quad - \frac{\pi i\overline{\vartheta}|\tau'|^2}{2\tau_2^3}(u_0 - \overline{u}_0)^2\partial_{\overline{u}_0} + \frac{i\lambda\tau_2}{\pi}\partial_{\overline{u}_0} + O(\vartheta)^2 \\
D_2 &= +\frac{2\pi i\overline{\vartheta}}{\tau_2}\partial_{\overline{a}_0} + \left(1 - \frac{\pi\vartheta\tau'}{\tau_2^2}\overline{u}_0\right)\partial_{u_0} - \frac{\pi\overline{\vartheta}\overline{\tau}'}{\tau_2^2}(u_0 - \overline{u}_0)\partial_{\overline{u}_0}.
\end{aligned}
$$

We can now find a modified complex structure on the K_3 such that Ξ will be a holomorphic curve. This means that in terms of the modified complex structure, $J_k^l + \delta J_k^l$, the equations $D_1\Xi = D_2\Xi = 0$ should imply $(\delta_k^l - iJ_k^l - i\delta J_k^l)\partial_l\Xi = 0$. In other words, $\partial_{u_0} - \frac{i}{2}\delta J_{u_0}^l\partial_l$ and $\partial_{z_0} - \frac{i}{2}\delta J_{z_0}^l\partial_l$ should be local linear combinations of D_1 and D_2. We can therefore calculate:

$$
\begin{aligned}
\delta J_{a_0}^{\overline{a}_0} &= \frac{2\pi i\overline{\vartheta}\tau'}{\tau_2^2}(u_0 - \overline{u}_0), \\
\delta J_{a_0}^{\overline{u}_0} &= -\frac{\pi\overline{\vartheta}|\tau'|^2}{\tau_2^3}(u_0 - \overline{u}_0)^2 + 2\lambda\tau_2, \\
\delta J_{u_0}^{\overline{a}_0} &= -\frac{4\pi\overline{\vartheta}}{\tau_2}, \\
\delta J_{u_0}^{\overline{u}_0} &= -\frac{2\pi i\overline{\vartheta}\overline{\tau}'}{\tau_2^2}(u_0 - \overline{u}_0).
\end{aligned}
$$

$$(20)$$

24

It is easy to check that δJ_k^l is invariant under $u_0 \to u_0 + m + n\tau(a_0)$.

We can also calculate the modified covariantly constant $(2,0)$ form. Writing it as $\omega + \delta\omega$, we find:

$$
\begin{aligned}
\delta\omega_{a_0\bar{a}_0} &= -\frac{i}{2}\delta J_{\bar{a}_0}^{u_0} = \frac{i\pi\bar{\vartheta}|\tau'|^2}{2\tau_2^3}(u_0 - \overline{u}_0)^2 - i\overline{\lambda}\tau_2, \\
\delta\omega_{u_0\overline{u}_0} &= \frac{i}{2}\delta J_{\overline{u}_0}^{a_0} = -\frac{2\pi i\vartheta}{\tau_2}, \\
\delta\omega_{u_0\bar{a}_0} &= \frac{i}{2}\delta J_{\bar{a}_0}^{a_0} = -\frac{\pi\vartheta\overline{\tau}'}{\tau_2^2}(u_0 - \overline{u}_0), \\
\delta\omega_{a_0\overline{u}_0} &= -\frac{i}{2}\delta J_{\overline{u}_0}^{u_0} = -\frac{\pi\vartheta\tau'}{\tau_2^2}(u_0 - \overline{u}_0).
\end{aligned}
$$

We can calculate:

$$
\int_{F'}\delta\omega = -2\pi i\vartheta, \qquad \int_{B'}\delta\omega = -i\overline{\lambda}. \tag{21}
$$

We can now find the requirement on λ in (18) such that a curve in the homology class:

$$
[\Sigma] = k[F'] + q[B]',
$$

will be analytic in the complex structure given by (20):

$$
0 = \int_{[\Sigma]}\delta\omega = -2\pi ik\vartheta - iq\overline{\lambda}.
$$

This implies:

$$
\lambda = -\frac{2\pi k}{q}\overline{\vartheta} + O(\theta)^2.
$$

5   Relation with little-string theories

Instantons on a noncommutative $\mathbf{T}^4$ are the solution to the Coulomb branch moduli space of certain 2+1D theories with $\mathcal{N} = 4$ supersymmetry [23].

In the following sections we will make an analogous statement about instantons on a noncommutative K_3. We will argue that the moduli space of instantons on a noncommutative K_3 provides the low-energy description of a certain 2+1D theory with $\mathcal{N} = 4$ supersymmetry. In general, the low-energy description of such theories is a supersymmetric σ-model on a hyper-Kähler manifold. The dimension of the manifold is the number of low-energy bosonic fields in the theory.

The theories that we consider are 5+1D theories compactified on $\mathbf{T}^3$. These 5+1D theories are the two heterotic little-string theories (LST) defined in [25] as the decoupled theories on NS5-branes of the heterotic string theory in the limit of zero coupling constant.

Let us take M_s to be the string-scale (the scale of the LST) and take R_1, R_2, R_3 to be the radii of the $\mathbf{T}^3$. For simplicity, we assume that the $\mathbf{T}^3$ is of the form $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$.

The heterotic LSTs have $\mathcal{N} = (1,0)$ supersymmetry. They also have an $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$ global symmetry inherited from the string-theory. In addition, they possess a global $Spin(4) = SU(2)_L \times SU(2)_R$ symmetry corresponding to rotations in directions transverse to the NS5-branes. The $SU(2)_R$ does not commute with the supersymmetry generators and is therefore an R-symmetry. The $SU(2)_L$ subgroup commutes with the SUSY generators and is an additional global symmetry.

When we compactify on $\mathbf{T}^3$ we have to specify the Wilson lines corresponding to the global symmetries. Since we wish to preserve supersymmetry we will not put any Wilson lines for the R-symmetry. We therefore have to specify $3 \times (16+1)$ Wilson line parameters.[1] We will refer to the 3 global $SU(2)_L$ Wilson line parameters as the α-**twists**.

5.1 Gauge theory limits

There are various limits of the parameters R_i for which the problem reduces to an ordinary gauge-theory question. We can obtain a guage theory limit by using one of two facts:

- The low-energy description of the 5+1D $Spin(32)/\mathbf{Z}_2$ LST is given by an $Sp(k)$ gauge theory with a global $Spin(32)/\mathbf{Z}_2$ symmetry and hyper-multiplets in the $(\mathbf{2k}, \mathbf{32})$ representation of $Sp(k) \times Spin(32)$ and the anti-symmetric $(\mathbf{k(2k-1)}, \mathbf{1})$ representation [49].

- For $k = 1$, the low-energy limit of the 5+1D E_8 CFT compactified on $\mathbf{T}^2$ is in general described by a strongly coupled 3+1D CFT [50]. For appropriately chosen E_8 Wilson lines we can get $SU(2)$ QCD with $N_f = 0, \ldots, 8$ ([51, 50]). For $k > 1$ and appropriately chosen Wilson lines we get the same $Sp(k)$ theory as above [52].

---

[1]We will not discuss compactifications without vector structure as in [46, 47, 48].

We can use this to generate 3+1D gauge theories as follows. Compactifying the 5+1D $Spin(32)/\mathbf{Z}_2$ LST on $\mathbf{S}^1$ to 4+1D we take $M_s \to \infty$ with $M_s R_1 \to \infty$ and $R_1 \to 0$. We can pick a Wilson-line $W \in Spin(32)/\mathbf{Z}_2$ such that a field in the fundamental $\mathbf{32}$ will have anti-periodic boundary conditions. We can combine it with a global $SU(2)_L$ Wilson-line (the "twists" discussed at the end of the previous subsection) such that all hypermultiplets get an extra $(-)$ factor in the boundary condition. In this way we can get a 4+1D $Sp(k)$ gauge theory with either some $\mathbf{2k}$ hypermultiplets (if we turn on both W and the twist) or a $\mathbf{k(2k-1)}$ hypermultiplet (if we just turn W on) or both (if we turn neither W nor the twist on) or none (if we turn on only the twist)! By controlling the value of the Wilson lines we can give small masses to any of these hypermultiplets. We can now compactify to 3+1D or 2+1D.

In [23], the equivalence of the moduli space on the Coulomb branch of the twisted compactified type-II little-string theory and the moduli space of noncommutative instantons on $\mathbf{T}^4$ was argued by embedding the NS5-branes inside a Taub-NUT space and then using a duality transformation to map the system to a system of D6-branes and a D2-brane. We will now use a similar technique, adapted to the heterotic theory.

5.2   Embedding in a Taub-NUT space

A Taub-NUT space is a circle-fibration over $\mathbf{R}^3$ such that the radius of the circle shrinks to zero at the origin. The metric is:

$$ds^2 = \rho^2 U(dy - A_i dx^i)^2 + U^{-1}(d\vec{x})^2, \qquad i = 1\ldots 3, \qquad 0 \le y \le 2\pi.$$

where ρ is the radius of the circle at $|\vec{x}| = \infty$ and,

$$U = \left(1 + \frac{\rho}{2|\vec{x}|}\right)^{-1}.$$

A_i is the gauge field of a monopole centered at the origin.

The property of the Taub-NUT space that is important for us is that the origin $|\vec{x}| = 0$ is a smooth point and is a fixed-point of the $U(1)$ isometry $y \to y + \epsilon$. This $U(1)$ isometry acts linearly on the tangent-space at the origin. In fact, it acts as a $U(1) \subset SU(2)_L$ subgroup of the rotation group $Spin(4) = SU(2)_L \times SU(2)_R$ that acts on the tangent space at the origin. The $(-1) \in Spin(4)$ that maps to the identity in $SO(4)$ corresponds to $y \to y + \pi$.

Now, if we place q NS5-branes at the center of the Taub-NUT space and take ρ to be very large, we can realize an α-twist as a shift $y \to y + \alpha$. When we compactify on $\mathbf{S}^1$ with coordinate $0 \le \theta \le 2\pi$, we can realize an $e^{i\alpha} \in U(1) \subset SU(2)_L$ twist, by identifying:

$$(\theta, \vec{x}, y) \sim (\theta + 2\pi, \vec{x}, y + \alpha).$$

This is a Dehn twist of the y-circle over the θ-circle. It can be set as a boundary condition at $|\vec{x}| = \infty$.

Now let us compactify on $\mathbf{T}^3$ with 3 α-twists. In this setting, we take a Taub-NUT space and q NS5-branes. Let us denote the coordinates as follows:

Object	0	1	2	3	4	5	6	7	8	9
Taub-NUT:	-	-	-	-	-	-	y	x_1	x_2	x_3
NS5:	-	-	-	-	-	-				
T^3:				-	-	-				

Here '-' denotes a direction which the object fills and y denotes the Taub-NUT circle direction. The α-twists become Dehn-twists of the 6th circle along the 3rd, 4th and 5th direction, just like [23].

Now let us consider the sphere $|\vec{x}| = R$ for $R \to \infty$. The circle in the 6th direction is fibered over it nontrivially and the $\mathbf{T}^3$ is fibered over it trivially. Let us combine the 6th circle to the $\mathbf{T}^3$ to form a $\mathbf{T}^4$. This $\mathbf{T}^4$ is fibered over the sphere $|\vec{x}| = R$ and we can adiabatically use the S-duality between heterotic string theory on $\mathbf{T}^4$ and type-IIA on K_3 to replace the background with a type-IIA background that at infinity looks like a K_3 fibered over the sphere $|\vec{x}| = R$. The Taub-NUT and NS5-branes in the original heterotic theory become other BPS objects at the center of the space in the type-IIA theory. By analyzing what the charge that corresponds to the Taub-NUT and NS5-branes transforms into under S-duality, we can determine what these objects are. Moreover, the α-twists map to some fluxes in the type-IIA theory. Our goal now is to determine what these fluxes are and what are the objects that the Taub-NUT and NS5-branes turn into.

Recall, that if one starts from k NS5-branes inside a q-centered Taub-NUT, then the total NS5-flux is equal to $k - q$. To see this one should recall the Bianchi Identity for 3-form field-strength H:

$$\frac{1}{2\pi} dH = -\frac{1}{8\pi^2} tr R \wedge R + k \delta(\vec{x}) \delta(y),$$

where we did not write the contribution from gauge fields since they are present in our consideration only as Wilson lines. Integrating this equation over S^3 at infinity one gets $\frac{1}{2\pi}\int_{S^3}H = k - q$.

5.3 Using the S-duality: $IIA/K_3 \Leftrightarrow H/\mathbf{T}^4$

Let us consider the $\mathbf{T}^4$ (in directions $3,4,5,6$) that is fibered over the sphere $|\vec{x}| = R$ at $R \to \infty$. Let us take 3 α-twists denoted by α_m ($m = 1\ldots3$). The metric on $\mathbf{T}^4$, written in block form with the blocks of size 1 and 3, is:

$$G_{kl} = \begin{pmatrix} R^2 & -\alpha_m R^2 \\ -\alpha_n R^2 & g_{nm} + \alpha_n \alpha_m R^2 \end{pmatrix}, \qquad G^{kl} = \begin{pmatrix} \frac{1}{R^2} + \alpha^k \alpha_k & \alpha^p \\ \alpha^m & g^{mp} \end{pmatrix} \tag{22}$$

Here $k,l = 6,3,4,5$. g_{mn} ($1 \le m,n \le 3$) is the metric on the $\mathbf{T}^3$ in directions $3,4,5$. R is the radius of the 6th direction and $\alpha^n = g^{nm}\alpha_m$. The Lagrangian for vector fields in $R^{2,1} \times R^3$ has the form

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}^i{}_{\mu\nu}(M^{-1})_{ij}\mathcal{F}^{j\mu\nu}.$$

Here $i = 1\ldots24$ and $M \equiv \Omega\Omega^T$ is the 24×24 symmetric matrix that specifies the point Ω in the moduli space: $SO(4,20)/SO(4) \times SO(20)$. In terms of the physical parameters, the matrix M is given in [53]:

$$M_{Het} = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}A^T \\ -C^T G^{-1} & G + C^T G^{-1}C + A^T A & C^T G^{-1}A^T + A^T \\ -AG^{-1} & AG^{-1} + A & \Gamma + AG^{-1}A^T \end{pmatrix}. \tag{23}$$

It is written in block form, with the blocks of sizes $4 + 4 + 16$. Here A_k^a ($k = 6,3,4,5$ and $a = 1\ldots16$) are the $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$ Wilson lines along the kth cycle of the $\mathbf{T}^4$. Γ is the Cartan matrix of $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$ and:

$$C_{kl} \equiv \frac{1}{2}A_k \cdot A_l + B_{kl} \tag{24}$$

with $A_k \cdot A_l \equiv A_k{}^b \Gamma_{bc} A_l{}^c$ and B_{kl} is the NSNS B-field.

We will also need to relate a point in the moduli-space, $\Omega \in SO(4,20)/SO(4) \times SO(20)$ to the physical parameters of type-IIA on K_3. The 24×24 matrix $M_{IIA} = \Omega\Omega^T$, is given in

[54] as:

$$\begin{pmatrix} e^\rho & -\frac{1}{2}e^\rho(b^I b^J d_{IJ}) & e^\rho b^I \\ -\frac{1}{2}e^\rho(b^I b^J d_{IJ}) & e^{-\rho} + b^I b^J d_{IK} H^K{}_J + \frac{1}{4}e^\rho(b^I b^J d_{IJ})^2 & -b^K H^I{}_K - \frac{1}{2}e^\rho b^I (b^K b^L d_{KL}) \\ e^\rho b^J & -b^K H^J{}_K - \frac{1}{2}e^\rho b^J (b^K b^K d_{KL}) & H^I{}_K d^{JK} + e^\rho b^I b^J \end{pmatrix}$$

(25)

The matrix is written in blocks of sizes $1 + 1 + 22$. $e^{-\rho}$ is defined as $e^{-\rho} = M_s{}^4 V_{K3}$, where M_s is the string scale of the IIA-theory and V_{K3} is the volume of K3. The vector b^I ($I = 1 \ldots 22$) specifies the integral of the NSNS 2-form over a basis of the 22-dimensional homology $H_2(K_3)$. Choosing a dual basis $\{\omega_I\}_{I=1}^{22}$ of $H^2(K_3, \mathbf{Z})$, we can write $B^{NS} = \sum b^I \omega_I$. The basis of $H_2(K_3, \mathbf{Z})$ is chosen such that the intersection matrix is (we show only non-zero entries):

$$d_{IJ} = \begin{pmatrix} & I_3 & \\ I_3 & & \\ & & \Gamma_{bc} \end{pmatrix}$$

(26)

The parameters $H^I{}_J$ describe the metric by specifying the splitting of $H^2(K_3)$ into self-dual and anti-self-dual parts. A 2-form $\sum_J \lambda^I \omega_I$ is self-dual if $\sum_J H^I{}_J \lambda^J = \lambda^J$ and anti-self-dual if $\sum_J H^I{}_J \lambda^J = -\lambda^J$. The $H^I{}_J$'s are constrained by the relations:

$$H^I{}_J H^J{}_K = \delta^I{}_K, \qquad d_{IJ} H^J{}_K = d_{KJ} H^J{}_I, \qquad H^J{}_I d_{JK} H^K{}_L = d_{IL},$$

so that $H^I{}_J$ has 57 independent parameters.

The matrix, M, satisfies:

$$M^T = M, \qquad M L M^T = L^{-1}$$

where

$$L = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & d_{IJ} \end{pmatrix} \qquad I = 1, \ldots 22.$$

An S-duality map is given by a 24×24 matrix, S, such that the point in moduli space in the type-IIA theory is given by:

$$M_{IIA} = S M_{Het} S^T.$$

In principle, S, is known up to an $SO(4, 20, \mathbf{Z})$ T-duality transformation.

5.4 The Taub-NUT and NS5-brane

We chose the matrix S that describes the duality transformation $M_{IIA} = S M_{Het} S^T$ so as to map the $k - q$ units of NS5-charge to $k - q$ $\widetilde{D6}$-branes and map q units of TN-flux to q D2-branes. We use the symbol $\widetilde{D6}$ to denote an object carrying one unit of $\int_{K_3} F \wedge F$ and having only pure D6-brane charge as measured at infinity (and no $H = dB$ flux). $\widetilde{D6}$-branes can be decomposed into a D6-brane and a D2-brane, where now D6 has zero $\int F \wedge F$ flux. D6-branes are wrapped over K_3 and D2-branes are points on K_3. We will consider the limit of shrinking the K_3 to zero volume and perform T-duality on the K_3 that interchanges D2 and $\widetilde{D6}$-branes.[2] After T-duality the moduli space is that of k instantons in $U(q)$, since:

$$(k - q)D2 + q\widetilde{D6} = kD2 + qD6.$$

5.5 The α-twists and instantons on a noncommutative K_3

The 80 parameters that specify the K_3 and the B^{NS} fluxes on the type-IIA side are functions of the parameters of the $\mathbf{T}^4$ compactification on the heterotic side. We now wish to know how the K_3 parameters depend on the relevant parameters on the heterotic side, namely, the α-twists and the metric on $\mathbf{T}^3$ (in directions $3, 4, 5$).

We find that the moduli of the metric on K_3 (i.e. $H_I{}^J$ and $e^{-\rho}$) are independent of the α-twists. Specifically:

$$
\begin{aligned}
& H^m{}_n = -g^{np} C_{pm}, && H^{m+3}{}_n = g^{nm}, && H^d{}_n = -A_l{}^{d-6} g^{ln}, \\
& H^m{}_{n+3} = g_{nm} + A_n \cdot A_m + C_{pn} g^{pk} C_{km}, && H^{m+3}{}_{n+3} = -C_{pn} g^{pm}, && H^d{}_{n+3} = A_n{}^{d-6} + A_l{}^{d-6} g^{lp} C_{pn}, \\
& H^m{}_c = A_{m,c-6} + C_{pm} g^{pk} A_{k,c-6}, && H^{m+3}{}_c = -A_{p,c-6} g^{pm}, && H^d{}_c = \delta_c^d + A_n{}^{d-6} g^{nm} A_{m,c-6}, \\
& m = 1 \ldots 3, && m + 3 = 4 \ldots 6, && d = 7 \ldots 24.
\end{aligned}
\tag{27}
$$

Here we used the notation of subsection (5.3). Also:

$$e^{-\rho} = \frac{1}{m_s{}^2 R^2}. \tag{28}$$

We can also check that the matrix S transforms the periodic parameters α_m (the twists), on the Heterotic side into the periodic moduli b^m (B^{NS} fluxes) on IIA side. In the basis of

---

[2]We are grateful to S. Sethi for useful discussions on this point.

$H_2(K_3, \mathbf{Z})$ that corresponds to (26), we have

$$b^m = \alpha_m \, (m = 1, 2, 3), \qquad b^{m+3} = 0 \, (m + 3 = 4, 5, 6), \qquad b^c = 0 \, (c = 7 \dots 22). \tag{29}$$

We will now argue that switching on α_m results in the instantons being noncommutative.

To make this point, let us also recall that in [1], noncommutative $U(N)$ SYM on $\mathbf{T}^4$ was defined by taking N D0-branes in $\mathbf{T}^4$ with B^{NS} flux and shrinking the area of $\mathbf{T}^4 \to 0$. Analogously, we can take a K_3 with B^{NS} flux and q D0-branes and we believe that in the limit that the K_3 shrinks to zero size the system is described by $U(q)$ NCG on a noncommutative K_3.

After S-duality we have the system of q D2-branes and $k - q$ $\widetilde{D6}$-branes. In order to study the moduli space of the system, we will discuss a system of q D0-branes and $k - q$ $\widetilde{D4}$-branes on K_3, where $\widetilde{D4}$-branes are defined to have only pure D4-charge.

Let us define $\tilde{b}^i$ (for $i = 1, 2, 3$) to be the components of B^{NS} along self-dual 2-forms in the decomposition into selfdual and anti-selfdual parts. We can show that a non-zero $\tilde{b}^i$ flux results in a bound state with mass:

$$m^2_{D0 + \widetilde{D4}} < (m_{D0} + m_{\widetilde{D4}})^2, \tag{30}$$

Here $m^2_{D0 + \widetilde{D4}}$ denotes the square of the mass of the bound state of q D2-branes and N $\widetilde{D4}$-branes, and $m_{\tilde{D4}}$ (m_{D0}) is the mass of the N $\widetilde{D4}$ (q D0) isolated branes.

Equation (30) means that one cannot separate the D0 and $\widetilde{D4}$-branes in the presence of a non-zero $\tilde{b}^i$. This in turn implies that instantons cannot shrink to zero size, which is a sign of noncommutativity.

We will prove (30) for arbitrary numbers q of D0-branes and N of $\widetilde{D4}$-branes. We start from the fact that N $\widetilde{D4}$-branes and q D0-branes are characterized by a Mukai vector $v = (N, 0, N - q)$, where the intersection product was defined as [55]:

$$v \cdot v' = \int \left(v^2 \wedge v'^2 - v^0 \wedge v'^4 - v'^0 \wedge v^4 \right), \qquad v = \left(v^0, v^2, v^4 \right), v^k \in H^k(K_3). \tag{31}$$

As it was done in [56] and [55], we defined vectors that span positive 4-plane in $H^*(K_3)$

$$E_0 = \left(1, B, \frac{1}{2} B \wedge B - \frac{1}{2} \sum_i \omega_i \wedge \omega_i \right), \qquad E_i = (0, \omega_i, B \wedge \omega_i) \, (i = 1, 2, 3). \tag{32}$$

Here ω_i $(i = 1, 2, 3)$ is a basis for the self-dual subspace of the 22-dimensional $H^2(K_3)$. We can find the projection of v onto this plane, v_{proj}, and determine the squared mass by $m^2 = v^2_{proj}$. Expanding the self-dual 2-forms as $\omega_i = \sum_{J=1}^{22} \epsilon^J_{(i)} \omega_J$, we find:

$$\int \omega_i \wedge \omega_j = d_{IJ} \epsilon^I_i \epsilon^J_j \tag{33}$$

where

$$\epsilon^J_{(i)} = \begin{pmatrix} -(C_{kn} + g_{kn})\delta^k_{(i)} \\ \delta^n_{(i)} \\ -A_k{}^b \delta^k_{(i)} \end{pmatrix} \tag{34}$$

and the relation between heterotic and IIA moduli was given in (27). From (32) and (34) it follows that

$$E_0 \cdot E_0 = \lambda, \qquad E_0 \cdot E_j = 0, \qquad E_i \cdot E_j = 2g_{ij}. \tag{35}$$

with $\lambda = 2\mathrm{tr}\{g\}$. Finally, we obtain:

$$m^2{}_{D0+\tilde{D}4} = \frac{1}{\lambda}\left(\frac{N}{2}\left(2 - \lambda + 2\tilde{b}g\tilde{b}\right) - q\right)^2 + 2N^2 \tilde{b}g\tilde{b}, \tag{36}$$

Here $\tilde{b}g\tilde{b} \equiv \tilde{b}_i g^{ij} \tilde{b}_j$ for $i, j = 1, 2, 3$. It is straightforward to show that $m^2{}_{D0+\tilde{D}4} < (m_{D0} + m_{\tilde{D}4})^2$ for any non-zero $\tilde{b}_i$ and any numbers N, q of D4 and D0-branes.

5.6 Generalized twists in the Heterotic LST

As pointed out in [25], the little-string theories exhibit T-duality. Compactification of the heterotic LSTs on $\mathbf{T}^d$ is specified by an external parameter space of $SO(d, 16 + d)/SO(d) \times SO(16 + d)$ (the metric, B-field, and Wilson lines) and there are discrete T-duality identifications given by $SO(d, 16 + d, \mathbf{Z})$ which act on the spectrum by exchanging momenta and winding quantum numbers (and can also mix it with global $U(1)^{16}$ quantum numbers which is the generic unbroken part of the gauge group). One can ask what happens if we compactify with an α-twist. In [23] it was argued that the T-dual of an α-twist is another kind of twist. For the T-dual "η-twist", a state with a nonzero $SU(2)_L$ charge also has a fractional winding number. We would now like to define generalized twists for the heterotic LSTs, in a similar manner.

In (27-28), we saw that the moduli of the metric on K_3 are independent of the values of the α-twists on the heterotic side. This fact suggests the following definition of a T-dual of

the α twist. Start from type-IIA using the moduli for the metric on K_3, given by equations (27-28), and take all components of B^{NS} (in the integer basis of $H^2(K_3, \mathbf{Z})$) to be non-zero $b^I = (\alpha_m, \eta^m, \gamma^c)$. Then, apply the transformation, given by S^{-1}, to the Heterotic theory and find the appropriate moduli.

For example, we can see that, with this definition of the generic twists, the $SU(2)_L$ Wilson lines along the $\mathbf{T}^3$ (on which the heterotic LST is compactified) are given, in terms of $\alpha_m, \eta^m, \gamma^c$, by:

$$G^{1m} = \alpha_k g^{mk} - \eta^n g^{mk} C_{kn} - \gamma^d g^{mk} a_{kd} + \frac{1}{2} e^\rho \eta^m (b^I d_{IJ} b^J), \qquad G^{nm} = g^{nm} + e^\rho \eta^n \eta^m, \quad (37)$$

The other parameters are rather complicated expressions and we present them in Appendix (A).

The point in moduli-space, on the type-IIA side, depends only on the self-dual part of the B-field, which is specified by 3 parameters. In subsection (5.5), we introduced the notation $\tilde{b}^i$ (for $i = 1, 2, 3$) for the components of the B-field along some fixed basis of the self-dual 2-forms on K_3. We are mostly interested in $\tilde{b}^i$ for $i = 1, 2, 3$, since they affect the moduli space of instantons. For the situation with generalized twists, they are given by:

$$\tilde{b}^i = \frac{1}{2} g^{in} \left(-\alpha_n + (C_{nk} + g_{nk}) \eta^k + A_n \cdot \gamma \right) \qquad (38)$$

5.7   Conclusion

We conjecture that the moduli space of the heterotic little-string theory of k NS5-branes compactified on $\mathbf{T}^3$ with α-twists is given by the moduli space $\mathcal{M}_{k,1}$ of k $U(1)$ instantons on a noncommutative K_3. The noncommutativity, θ, is specified by a symplectic form whose inverse, θ^{-1} is a closed 2-form on the K_3. Its expansion in terms of an integral basis of $H^2(K_3, \mathbf{Z})$ is given in terms of the 3 α-twists in (29).

6   Generalization to NS5-branes at A_{q-1} singularities

In previous sections we found moduli spaces of instantons of a noncommutative $U(1)$ gauge theory. How can we get instantons for $U(q)$ gauge theories with $q > 1$? On the face of it, all we need to do is replace the Taub-NUT space with a multi-centered Taub-NUT space. In

particular, the classical multi-centered Taub-NUT space can have, for a particular choice of parameters, an A_{q-1} singularity and we can naively conclude that the moduli space of the theories that we get by placing heterotic NS5-branes at A_{q-1} singularities and compactifying on $\mathbf{T}^3$ is the moduli space of $U(q)$ instantons on K_3 as before.

However, unlike the type-II string theories, the moduli spaces of A_{q-1} singularities and NS5-branes in A_{q-1} singularities, receive quantum corrections [57, 58, 59, 60]. For example, the hypermultiplet moduli space of an A_1 singularity, corresponding to the normalizable metric and 2-form deformations, in the heterotic string theory is a blow-up of $\mathbf{R}^4/\mathbf{Z}_2$ [57, 58] and does not have a singular point that can be associated with a singular space, even though classically the moduli space is exactly $\mathbf{R}^4/\mathbf{Z}_2$ and the origin of it is singular and can be associated with a singular space.

We would like to generalize the question we asked in (5) to a question like: "what is the Coulomb branch moduli space of the 5+1D theory of k NS5-branes at an A_{q-1} singularity compactified on $\mathbf{T}^3$ with Wilson lines." In the type-II LST case, the answer was proposed in [23] to be the moduli space of k noncommutative $U(q)$ instantons on $\mathbf{T}^4$.

In the heterotic case, the statement of the problem is somewhat ambiguous since we need to specify at what point in the hyper-multiplet moduli space we are. Let us denote this hyper-multiplet moduli space by $\mathcal{Y}$. If there is no singularity in $\mathcal{Y}$, then there is no implied "special" point as in the type-II case.

We can replace the A_{q-1} singularity with a Taub-NUT space. Then, the Taub-NUT space has a $U(1)$ isometry which, as before, becomes a global symmetry of a decoupled theory on the NS5-branes. This $U(1)$ also acts nontrivially on the hyper-multiplet moduli space $\mathcal{Y}$. If we put nonzero Wilson lines of this $U(1)$ along the $\mathbf{T}^3$ (the "twists") then we are forced to be at a fixed-point of this $U(1)$ in $\mathcal{Y}$. This restricts the choice of points in $\mathcal{Y}$, but there are still several cases. We will argue below that these match nicely with properties of the moduli space of instantons on K_3.

6.1   Review of heterotic NS5-branes at singularities

The type of 5+1D low-energy that one gets for k NS5-branes and an A_{q-1} singularity of the heterotic string theories, was analyzed in [61]. They found that for $k < 4$ the low-

energy description is the same as for $q = 1$ for both $E_8 \times E_8$ as well as $Spin(32)/\mathbf{Z}_2$ gauge groups. They also characterized the theories in terms of the local gauge group G and the number of tensor multiplets, n_T. They found that k E_8 NS5-branes always give $n_T \geq k$ and k $Spin(32)/\mathbf{Z}_2$ NS5-branes always give $Sp(k)$ as a factor of G. For $q \geq 2$ and $k = 4$ E_8 NS5-branes one always has $G = SU(2)$ and $n_T = 4$. For $q \geq 2$ and $k = 4$ $Spin(32)/\mathbf{Z}_2$ one has $G = Sp(4)$ and $n_T = 1$.

6.2   The hypermultiplet moduli space

The hyper-multiplet moduli space, $\mathcal{Y}$, of k NS5-branes at a q-centered Taub-NUT space comprises, classically, of the positions of the NS5-branes ($4k$ variables), the blow-up modes of the metric on the Taub-NUT space ($3(q - 1)$ parameters) and the NSNS 2-form modes ($(q-1)$ parameters). As explained in [57, 58], it receives quantum corrections that depend on α' but not on the string coupling constant λ. If we also include the absolute position of the Taub-NUT space we get $4(k+q)$ dimensions. It was argued in [59] that $\mathcal{Y}$ is the same as the Coulomb branch moduli space of 2+1D QCD with gauge group $SU(q) \times U(1)^k$ and $N_f = k$ massless quarks that are in the fundamental representation of $U(q)$ and are charged under one of the $U(1)$'s. The coupling constant of the $U(1)$'s was argued to be infinite. In fact, the duality that we are using is the same as that used by [59] to solve the hyper-multiplet moduli space.

This space has an overall $U(1)$ isometry. Physically, this $U(1)$ isometry is the isometry of the Taub-NUT space and acts on the positions of the NS5-branes. In the 2+1D QCD language, let us consider the diagonal $U(1)$ subgroup of $U(1)^k$. The 2+1D dual of the photon that corresponds to this $U(1)$ can be shifted without changing the metric on the Coulomb branch. This is the $U(1)$ isometry on the quantum space.

As we have argued, once we compactify on $\mathbf{T}^3$ and introduce generic twists, the hyper-multiplet moduli space reduces to the fixed point locus of the $U(1)$ isometry. Let us denote this locus by $\mathcal{Y}^{(\prime)}$. From the classical limit, We expect it to be of dimension $4q$. If we ignore the overall center of mass position of the Taub-NUT space, we get $4(q - 1)$ parameters that correspond to the deformation modes of the Taub-NUT solution. For example, with one NS5-brane ($k = 1$) and $q = 2$, the Taub-NUT space has two centers where the fibered circle

shrinks to zero. The NS5-brane can be placed in either of these centers. In terms of instantons, this would correspond to breaking $U(2) \to U(1) \times U(1)$ and placing the instanton in one of the $U(1)$ factors. If the instantons were commutative, this configuration would be supersymmetric. However, for noncommutative instantons this configuration has to break supersymmetry. One way to see this is in terms of curves in K_3. The classes $[B] + [F]$ and $[B]$ cannot be simultaneously analytic unless the K_3 is elliptically fibered with a section. In terms of the twisted generalized LST, this means that after compactification on $\mathbf{T}^3$ with generic twists, we expect that there is no supersymmetric vacuum.

With $k = 1$ and $q = 2$, the singularity in the hyper-multiplet moduli space is smoothed out by quantum corrections and that means that we cannot realize instantons with $k = 1$ inside $U(2)$ except by breaking $U(2) \to U(1) \times U(1)$. In terms of holomorphic curves it means that we cannot find a holomorphic curve of class $2[B] + [F]$ inside K_3.

In order to realize instantons inside $U(2)$, we have to have a special point in $\mathcal{Y}^{(\prime)}$ where intuitively, the two centers of the Taub-NUT coincide. This means that the moduli space should have a singularity. Since the moduli space of $SU(2)$ QCD with $N_f = k$ is singular for $k \geq 2$, we see that we expect to have curves of classes $2[B] + k[F]$ for $k \geq 2$, inside the K_3. This is also the condition that the self-intersection of the curve should be greated or equal to (-2) which is required for irreducible curves. This also agrees [62] with the results obtained from the duality between type-IIA on K_3 and heterotic string theory on $\mathbf{T}^4$.

6.3 The Coulomb branch

After we have placed k NS5-branes at the q-centered Taub-NUT space and taken the decoupling limit $\lambda \to 0$ with M_s kept fixed, we obtain a 5+1D little-string theory with a low-energy description that, in general could contain tensor multiplets, vector multiplets and hyper-multiplets. We are interested in the Coulomb branch of the theory after compactification on $\mathbf{T}^3$ with twists of the global $U(1)$ discussed above. Following the same arguments as in section (5), we can conclude that the moduli space is the same as the moduli space of k instantons of noncommutative $U(q)$ Yang-Mills theory on K_3. The three parameters that specify the anti-self-dual part of the 2-form that determines the noncommutativity of the K_3 is determined, as before, by the twists.

7 Seiberg-Witten curves, spectral-curves and T-duality

As an application of these results we can motivate the spectral-curve construction of instanton moduli spaces on a noncommutative K_3.

We have seen that the moduli space $\mathcal{M}_{k,n}(\theta)$ of k noncommutative $U(n)$ instantons on K_3 with a noncommutativity parameter θ, given by an anti-self-dual 2-form on the K_3 corresponds to the moduli space of the Coulomb branch of a certain 5+1D theory compactified on $\mathbf{T}^3$.

Let us take a special $\mathbf{T}^3$ of the form $\mathbf{T}^2 \times \mathbf{S}^1$ and let us take the limit that the radius R, of $\mathbf{S}^1$, is very big. We can then compactify in two steps. The first step is to obtain a 3+1D theory by compactifying the 5+1D theory on $\mathbf{T}^2$. The low-energy of this theory is described by a certain Seiberg-Witten curve of genus g. When $R \to \infty$ we can compactify the low-energy effective 3+1D action on $\mathbf{S}^1$. This procedure was described in [21], and we get a a hyper-Kähler moduli space that is described as the collection of Jacobian varieties of all the Seiberg-Witten curves. Recall that all the Seiberg-Witten curves form a g-dimensional (complex) space and the Jacobian $\mathbf{T}^{2g}$, of each curve, is also a g-dimensional complex space. Together we get a $2g$-dimensional space which is hyper-Kähler.

In the context of instanton moduli spaces, the Seiberg-Witten curves are called the "spectral-curves" and the points on the Jacobian are called the "spectral-bundle."

As an example, let us describe how these considerations translate to the spectral-curve construction of noncommutative instantons on $\mathbf{T}^4$. According to [23], the moduli space $\mathcal{M}_{k,n}$ of instantons on a noncommutative $\mathbf{T}^4$ is equivalent to the moduli space of little string theories on $\mathbf{T}^3$ with twists. The $\mathbf{T}^2$ can always be written (in several ways) as a $\mathbf{T}^2$-fibration over a base $\mathbf{T}^2$. Let us denote the base by B and let z be a doubly-periodic coordinate on B such that $z \sim z + e_1$ and $z \sim z + e_2$. Let $w_a(z)$ $(a = 1 \ldots n)$ be the local map from the base to the fiber $\mathbf{T}^2$.

If we take the little string theories on $\mathbf{T}^2 \times \mathbf{S}^1$ with $\mathbf{S}^1$ very large, we get a $\mathbf{T}^2$-fiber that is very small. The little-string theories moduli space can also be deduced using the construction in [69] of nk D4-branes suspended between pairs of cyclic n NS5-branes. In this limit, the twists correspond to Dehn twists as in the elliptic models of [69].

Thus we can conclude that if the noncommutativity is described by a 2-form θ_{iI} with $i = 1, 2$ a coordinate on the base $\mathbf{T}^2$ and $I = 1, 2$ a coordinate on the fiber $\mathbf{T}^2$, then the spectral curves are N-fold maps from the base to the fiber with twisted boundary conditions given by:

$$w_a(z + \sum_{j=1}^{2} N_j e_j) = w_a(z) + \sum_{I=1}^{2} \theta_{iI} b_I.$$

Here b_1, b_2 are a basis of the lattice such that the fiber $\mathbf{T}^2$ is given by:

$$w \sim w + b_1 \sum w + b_2.$$

7.1 Relation to T-duality

The relation between moduli spaces of noncommutative instantons on $\mathbf{T}^4$ and K_3 and spectral curves also follows from T-duality of type-IIA on these spaces. This is a generalization of the commutative case [62], and it was explicitly constructed for $\mathbf{T}^4$ in [63]. Let us briefly recall how this works for K_3. We start with the definition of noncommutative $U(q)$ gauge theory on K_3, in the spirit of [1, 2]. Namely, we take q D0-branes and k D4-branes on K_3 and send the volume of K_3 to zero while keeping a constant NSNS B-field flux. We now perform a T-duality transformation that transforms the K_3 into another K_3 that we can present as an elliptic fibration (not necessarily with a section) and such that a D0-brane turns into a D2-brane wrapped on the fiber and a D4-brane turns into a D2-brane wrapped on the base. The base and fiber correspond to $H_2(K_3, \mathbf{Z})$ classes but they are not necessarily analytic in the same complex structure. Now the $k+q$ D2-branes form a single D2-brane that is analytic in some complex structure.

Let us describe the original K_3 (with the q D0-branes and k D4-branes) as in (32). We write it as:

$$E_0 = (1, 0, 0, \int_{C_3} B, \ldots, \int_{C_{22}} B, \int \frac{1}{2} B \wedge B - V), \qquad E_J = (0, S_1, S_2, \int_{C_3} J, \ldots, \int_{C_{22}} J, 0)$$

$$E_\omega = (0, 0, 0, \int_{C_3} \omega, \ldots, \int_{C_{22}} \omega, B \wedge \omega), \qquad E_{\overline{\omega}} = (0, 0, 0, \int_{C_3} \overline{\omega}, \ldots, \int_{C_{22}} \overline{\omega}, B \wedge \overline{\omega})$$

where V is the volume of K_3 and $C_1, \ldots, C_{22}$ is a basis for $H_2(K_3, \mathbf{Z})$, which we choose such that C_1 corresponds to the fiber of the elliptic fibration, and C_2 corresponds to the base.

The intersection numbers are:

$$C_1 \cdot C_2 = 1, \qquad C_1 \cdot C_1 = 0, \qquad C_2 \cdot C_2 = -2.$$

We used the fact that $\int_{C_1} \omega = \int_{C_2} \omega = 0$, (the fiber and the base of the original K_3 are analytic in the chosen complex structure) and that we are dealing with NSNS flux B having only (2,0) and (0,2) parts. We also introduced the notation for the volume of the fiber and the base respectively $S_1 \equiv \int_{C_1} J$, $\quad S_2 \equiv \int_{C_2} J$, with J being Kahler form on the K_3. After T-duality the vectors which span a positive-definite 4-plane in $R^{(4,20)}$ become:

$$
\begin{aligned}
'E_0 &= (0, 1, \int \frac{1}{2} B \wedge B - V, \int_{C_3} B, \ldots, \int_{C_{22}} B, 0), \\
'E_{J'} &= (S_1, 0, 0 \int_{C_3} J, \ldots, \int_{C_{22}} J, S_2), \\
'E_{\omega'} &= (0, 0, \int B \wedge \omega, \int_{C_3} \omega, \ldots, \int_{C_{22}} \omega, 0)
\end{aligned}
$$

After interchanging $'E_0$ and $'E_{J'}$ we can read off the following information:

$$
\begin{aligned}
\int_{C_1'} J' = 1 \quad & \int_{C_2'} J' = \tfrac{1}{2} \int B \wedge B \\
\int_{C_1'} \omega' = 0 \quad & \int_{C_2'} \omega' = \int B \wedge \omega, \\
\int_{C_1'} \overline{\omega'} = 0 \quad & \int_{C_2'} \overline{\omega'} = \int B \wedge \overline{\omega}.
\end{aligned}
$$

We want to find a complex structure on a dual K_3 such that curve $\Sigma = q[C_2'] + k[C_1']$ be analytic. So, we define a new form

$$\Omega = aJ' + b\omega' + c\overline{\omega'}$$

and impose three conditions:

$$\int_\Sigma \Omega = 0, \qquad \int_{K_3'} \Omega \wedge \Omega = 0, \qquad \int_{K_3'} \Omega \wedge \overline{\Omega} = \mathrm{Vol}(K_3').$$

The first condition results in

$$a \left(q \left(\frac{1}{2} \int B \wedge B - V \right) + k \right) + bq \int B \wedge \omega + cq \int B \wedge \overline{\omega} = 0.$$

At this point we take the limit of $V \to 0$ as in [1, 2, 3]. We obtain:

$$a \left(\frac{q}{2} \int B \wedge B + k \right) + bq \int B \wedge \omega + cq \int B \wedge \overline{\omega} = 0. \tag{39}$$

40

The second and third conditions give:

$$a^2 + 2bc = 0, \qquad |a|^2 + |b|^2 + |c|^2 = 1.$$

Now let us take the limit that θ is small and compare the $O(\theta)$ term found in (21) with the present calculation. We assume that the B-field on (the original) K_3 takes the form:

$$2\pi i B = \vartheta^{-1}\omega - \overline{\vartheta}^{-1}\overline{\omega}.$$

This equation is the analog of $\theta = B^{-1}$ in the flat space case of [3]. In order to be consistent with the conventions of [3] of assigning dimensions to parameters, namely that B and θ are dimensionless and only the metric is dimensionful, we have to assume that ω is dimensionless and is normalized such that $\int \omega \wedge \overline{\omega}$ is kept fixed. This implies that $a = O(\vartheta)$, $b = 1 + O(\vartheta)$ and $c = O(\vartheta)^2$. To check the relation between the NSNS flux B on the original K_3 an the noncommutativity parameter ϑ we compare $\int_{C_1'} \Omega = a$ with $\int_{F'} \delta\omega = -2\pi i\vartheta$ (obtained in (21)) and find agreement. Note that for small ϑ the period $\int_{F'} \delta\omega$ is independent of k and q. However, from equation (39) we expect higher order terms in ϑ to depend on k and q.

8   Summary and further directions

Let us summarize our results:

- We have explicitly constructed instantons on a noncommutative $\mathbf{T}^4$ in terms of spectral curves. The spectral curves are constructed inside a modified $\mathbf{T}^4$ given by (8).

- We wrote down the instanton equations on a noncommutative manifold, to first order in the noncommutativity. We have seen that for a generic metric there is an extra curvature dependent term that vanishes for hyper-Kähler manifolds.

- We have explicitly constructed instantons on a noncommutative K_3, to first order in the noncommutativity, in terms of spectral curves. The spectral curves are constructed inside a modified K_3 with a complex structure given by (20).

- We have argued that the moduli spaces of noncommutative $U(1)$ instantons on K_3 are the Coulomb branch moduli spaces of compactified heterotic little-string theories and the moduli spaces of noncommutative $U(q)$ instantons on K_3 are the Coulomb branch

41

moduli spaces of compactified generalized heterotic little-string theories obtained from heterotic NS5-branes at A_{q-1} singularities.

Let us suggest some further directions that might be interesting to study:

- Expand the instanton equations on a curved manifold to all orders in the noncommutativity parameter or the curvature and prove the spectral curve construction of instantons on a noncommutative K_3 directly (rather than from the T-duality argument).

- Explore the relation between the moduli spaces of noncommutative instantons on K_3 and observations of Aspinwall and Morrison [61] about the theories of heterotic NS5-branes at A_{q-1} singularities. For example, the observation that for $k < 4$ the low-energy description is the same as for $q = 1$ is related to the fact that for $k < 4$ and $q > 1$ the commutative instantons are reducible and therefore the moduli spaces of the theory compactified on $\mathbf{T}^3$ (without twists) decomposes into a product. (Recall from (4.1) that if $k < 2q$, we cannot find an irreducible curve.)

Acknowledgments

A   Appendix

In section 4 we have presented G^{mn} and G^{1m} parameters, see eq.(37), which correspond to U-dual of α-twist. Here we will complete the list of Heterotic moduli in the presence of the generic twist.

$$G^{11} = e^{-\rho} + b^I d_{IK} H^K{}_J b^J + \frac{1}{4} e^\rho (b^I d_{IK} b^K)^2, \tag{40}$$

where $H^K{}_J$ are taken from (28), d_{IJ} from (26) and $b^I = (\alpha_m, \eta^m, \gamma^c)$.

$$a'_{kb} = \mathcal{K}^{-1}_{kn} \Big(G^{11}(H^n{}_b + e^\rho \alpha_n \gamma_b) - G^{n1}(H^b{}_J b^J + \frac{1}{2} e^\rho \gamma^b b^I d_{IJ} b^J) \Big), \tag{41}$$

where $\mathcal{K}^{nk} = G^{n1}G^{1k} - G^{11}G^{nk}$. Let us draw your attention to the fact that in the presence of the generic twist Wilson lines over the circles of T^3 are different from what one has without twists. Only for α-twist $a'_{kb} = a_{kb}$.

In the generic case one has to introduce Wilson line over the Taub-Nut circle at infinity

$$a_1^b = -\frac{H^b{}_J b^J + \frac{1}{2}e^\rho \gamma^b b^I d_{IJ} b^J + G^{1n} a'_n{}^b}{G^{11}} \tag{42}$$

and switch on non-zero B_{n1} field

$$B_{n1} = (e^\rho \eta^m - \frac{1}{2}a_1 \cdot a_1 G^{1m})\mathcal{N}_{mn} - \frac{1}{2}a'_n \cdot a_1, \tag{43}$$

where $\mathcal{N}_{mn} = (G^{mn})^{-1}$.

One gets a complicated expression for the B'_{mn} field. It isalso different from the B_{mn} field, present before the twist:

$$B'_{kn} = -\mathcal{N}_{km}\left(H^n{}_m + e^\rho \alpha_n \eta^m + (B_{1n} + \frac{1}{2}a_1 \cdot a'_n)G^{1m}\right) - \frac{1}{2}a'_k \cdot a'_n \tag{44}$$

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